Hugh Montgomery · Ashkan Nikeghbali
Michael Th. Rassias *Editors*

Exploring the Riemann Zeta Function

190 years from Riemann's Birth

With a preface by
Freeman J. Dyson

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ISBN 978-3-319-59968-7 ISBN 978-3-319-59969-4 (eBook)
DOI 10.1007/978-3-319-59969-4

Library of Congress Control Number: 2017947901

Mathematics Subject Classification (2010): 11-XX, 30-XX, 33-XX, 41-XX, 43-XX, 60-XX

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The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface by Freeman J. Dyson: Quasi-Crystals and the Riemann Hypothesis

On the occasion of the 190 years from Riemann's birth, which led to the conception of this book, I am presenting a challenge to young mathematicians to study quasicrystals and use them to prove the Riemann Hypothesis. Among the readers, those who are experts in number theory may consider the challenge frivolous. Those who are not experts may consider it uninteresting or unintelligible. Nevertheless, I am putting it forward for your serious consideration. One should always be guided by the wisdom of the physicist Leo Szilard, who wrote his own Ten Commandments to stand beside those of Moses. Szilard's second commandment says, "Let your acts be directed toward a worthy goal, but do not ask if they will reach it; they are to be models and examples, not means to an end." The Riemann Hypothesis is a worthy goal, and it is not for us to ask whether we can reach it. I will give you some hints describing how it might be achieved. Here, I will be giving voice to the mathematician that I was almost 70 years ago before I became a physicist.

There were until recently two supreme unsolved problems in the world of pure mathematics, the proof of Fermat's Last Theorem and the proof of the Riemann Hypothesis. In 1998, my former Princeton colleague Andrew Wiles polished off Fermat's Last Theorem, and only the Riemann Hypothesis remains. Wiles' proof of the Fermat Theorem was not just a technical stunt. It required the discovery and exploration of a new field of mathematical ideas, far wider and more consequential than the Fermat Theorem itself. It is likely that any proof of the Riemann Hypothesis will likewise lead to a deeper understanding of many diverse areas of mathematics and perhaps of physics too. The Riemann Hypothesis says that one specific function, the zeta-function that Riemann named, has all its complex zeros upon a certain line. Riemann's zeta-function and other zeta-functions similar to it appear ubiquitously in number theory, in the theory of dynamical systems, in geometry, in function theory, and in physics. The zeta-function stands at a junction where paths lead in many directions. A proof of the hypothesis will illuminate all the connections. Like every serious student of pure mathematics, when I was young, I had dreams of proving the Riemann Hypothesis. I had some vague ideas that I thought might lead to a proof, but never pursued them vigorously. In recent years, after the discovery

of quasicrystals, my ideas became a little less vague. I offer them here for the consideration of any young mathematician who has ambitions to win a Fields Medal.

Now, I jump from the Riemann Hypothesis to quasicrystals. Quasicrystals were one of the great unexpected discoveries of recent years. They were discovered in two ways, as real physical objects formed when alloys of aluminum and manganese are solidified rapidly from the molten state and as abstract mathematical structures in Euclidean geometry. They were unexpected, both in physics and in mathematics. I am here discussing only their mathematical properties.

Quasicrystals can exist in spaces of one, two, or three dimensions. From the point of view of physics, the three-dimensional quasicrystals are the most interesting, since they inhabit our three-dimensional world and can be studied experimentally. From the point of view of a mathematician, one-dimensional quasicrystals are more interesting than two-dimensional quasicrystals because they exist in far greater variety, and the two-dimensional are more interesting than the three-dimensional for the same reason. The mathematical definition of a quasicrystal is as follows. A quasicrystal is a distribution of discrete point masses whose Fourier transform is a distribution of discrete point frequencies. Or to say it more briefly, a quasicrystal is a pure point distribution that has a pure point spectrum. This definition includes as a special case the ordinary crystals which are exactly periodic point distributions with exactly periodic point spectra.

Excluding the ordinary crystals, quasicrystals in three dimensions come in very limited variety, all of them being associated with the icosahedral rotation group. The two-dimensional quasicrystals are more numerous, roughly one distinct type associated with each regular polygon in a plane. The two-dimensional quasicrystal with pentagonal symmetry is the famous Penrose tiling of the plane, discovered by Penrose before the general concept of quasicrystal was invented. Finally, the one-dimensional quasicrystals have a far richer structure since they are not tied to any rotational symmetries. So far as I know, no complete enumeration of one-dimensional quasicrystals exists. It is known that a unique quasi-crystal exists corresponding to every Pisot-Vijayaraghavan number or PV number. A PV number is a real algebraic integer, a root of a polynomial equation with integer coefficients, such that all the other roots have absolute value less than one (see [1]). The set of all PV numbers is infinite and has a remarkable topological structure. The set of all one-dimensional quasicrystals has a structure at least as rich as the set of all PV numbers and probably much richer. We do not know for sure, but it is likely that a huge universe of one-dimensional quasicrystals not associated with PV numbers is waiting to be discovered.

Here comes the punch line of my argument, the essential point linking one-dimensional quasicrystals with the Riemann Hypothesis. If the Riemann Hypothesis is true, then the zeros of the zeta-function form a one-dimensional quasicrystal according to the definition. They constitute a distribution of point masses on a straight line, and their Fourier transform is likewise a distribution of point masses, one at each of the logarithms of ordinary prime-power numbers. Andrew Odlyzko (see [2]) has published a beautiful computer calculation of the Fourier transform

of the zeta-function zeros. The calculation shows precisely the expected structure of the Fourier transform, with a sharp discontinuity at every logarithm of a prime-power number and nowhere else.

My proposal is the following. Let us pretend that we do not know that the Riemann Hypothesis is true. Let us tackle the problem from the other end. Let us try to obtain a complete enumeration and classification of all one-dimensional quasicrystals. That is to say, we enumerate and classify all point distributions that have a discrete point spectrum. We shall then find the well-known quasicrystals associated with PV numbers and also a whole slew of other quasicrystals, known and unknown. Among the slew of other quasicrystals, we should be able to identify one corresponding to the Riemann zeta-function and one corresponding to each of the other zeta-functions that resemble the Riemann zeta-function. Suppose that we can prove rigorously that one of the quasicrystals in our enumeration has properties that identify it with the zeros of the Riemann zeta-function. Then we have proved the Riemann Hypothesis and can wait for the telephone call announcing the award of the Fields Medal.

These are of course idle dreams. The problem of classifying one-dimensional quasicrystals is horrendously difficult, probably at least as difficult as the problems that Andrew Wiles took 7 years to fight his way through. But still, the history of mathematics is a history of horrendously difficult problems being solved by young people too ignorant to know that they were impossible. The classification of quasicrystals is a worthy goal and might even turn out to be achievable. Problems of that degree of difficulty will not be solved by old men like me. Let the young people now have a try.

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An Introduction to Riemann's Life, His Mathematics, and His Work on the Zeta Function

Roger Baker

Abstract Although the zeta function was first defined and used by Euler, it is to Bernhard Riemann, in an article written in 1859, that we owe our view of the zeta function as a meromorphic function in the plane with a functional equation. Riemann is a very remarkable figure in the history of mathematics. The present article describes his career including the major mathematical highlights, and gives some discussion of his published and unpublished work on the zeta function.

1 Introduction

In 1846 a frail and painfully shy youth of nineteen arrived in Göttingen, in the Kingdom of Hanover, to begin his university studies. Bernhard Riemann was to spend most of his adult life in Göttingen as an undergraduate, doctoral candidate, Privatdozent (junior university instructor), extraordinary professor (roughly, associate professor), and full professor. The best of his research papers, written in the 1850s, were among the very most influential papers in the mathematics of his century. In particular, in a paper written in 1859, he set Euler's zeta function (as he would have referred to it) on a modern footing. Today, zeta functions are perhaps the part of number theory that we would most like to understand better, as indeed the publication of the present volume testifies.

Georg Friedrich Bernhard Riemann was born on 17 September 1826 in Breselenz, in the Kingdom of Hanover. His father Friedrich was the local pastor; later he took over the parish of Quickborn, not far from the Free Imperial City of Hamburg, but still in the Kingdom of Hanover. Bernhard was tutored by his father up to the age of thirteen, when he moved to the city of Hanover; he lived with his grandmother and attended high school for 2 years. After her death, he attended the high school in Lüneburg, which is south-east of Hamburg. He was not notably successful except in mathematics. (At this time he read Legendre's *Théorie des Nombres* [12].)

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H. Montgomery et al. (eds.), *Exploring the Riemann Zeta Function*,
DOI 10.1007/978-3-319-59969-4_1

On arriving in Göttingen, Riemann initially matriculated as a student of philosophy and theology, perhaps at the time thinking of following his father's profession. Certainly Riemann was very devout. But he soon focused on mathematics.

In the summer of 1846, Riemann took Moritz Stern's course on numerical solution of equations and Carl Goldschmidt's course on terrestrial magnetism. Stern (1807–1894) and Goldschmidt (1807–1851) were both Gauss's students. Stern was the first Jewish full professor in Germany and is known for the Stern–Brocot tree, with vertices corresponding to the positive rationals. Goldschmidt was Gauss's assistant at the University Observatory and also helped Gauss gather data comparing the prime number counting function

$$\pi(x) = \sum_{p \le x} 1$$

to the logarithmic integral

$$\operatorname{li} x = \int_0^x \frac{dt}{\log t}$$

(this is a Cauchy principal value). Carl Friedrich Gauss (1777–1855) was deservedly famous for many contributions to mathematics, geodesy, and physics. By this time the elderly Gauss, who had been at Göttingen since 1807, would only give classes on his method of least squares, which included introductory linear algebra. Riemann took this course in winter 1846–1847, along with Stern's course on definite integrals. Presumably in search of deeper mathematical content, Riemann spent the next 2 years, 1847–1849, at the University of Berlin. He attended Dirichlet's courses in number theory, the theory of definite integrals, and partial differential equations; Jacobi's lectures in analytical mechanics and linear algebra; and Eisenstein's lectures on the theory of elliptic functions.

We pause here to discuss the influence of these three fine mathematicians on Riemann. F.G.M. Eisenstein (1823–1852) worked mostly in number theory. Riemann recounted to Dedekind that they had discussed the use of complex numbers in the theory of functions, which seemed important to Riemann, and their views differed so radically that there was not much further interaction. C.G.J. Jacobi (1804–1851) was famous for his contributions to the theory of elliptic functions, which he based on four theta functions. He also worked on number theory and partial differential equations and dynamics. His collected works fill eight large volumes.

Riemann was certainly very interested in the work on elliptic functions, as we shall see. However, the key person for his evolution as a mathematician was P.G.L. Dirichlet (1805–1859). Dirichlet was a brilliant lecturer and some of his lectures on number theory are still extant [5]. He was regarded by Jacobi as the only fully rigorous analyst. His paper of 1829 on pointwise convergence of Fourier series was a model of precision; this would be the area of Riemann's habilitation thesis of 1854. His introduction of "Dirichlet characters" and the associated L functions

[treated as functions on $(0, \infty)$] were needed for his proof that there are infinitely many primes in any arithmetic progression $a \pmod q$ with $(a, q) = 1$; this was completed in a paper of 1839–1840. Dirichlet's wife Rebecka was the sister of the great Felix Mendelssohn-Bartholdy.

We mention briefly the political events of 1848 and beyond. At this time Germany was a loosely associated group of 38 states including Hanover; the most important were Prussia, with Berlin as the capital, and Austria, with Vienna as the capital. Many of these states and other parts of Europe were shaken by revolution in 1848–1849. Ultimately, the revolution in German states was put down by military force, but modernization of political systems followed gradually. In 1862, Otto von Bismarck became Prussian prime minister. His skills, and Prussian military prowess, led by 1871 to the unification of Germany as an empire excluding Austria–Hungary. Bismarck was the Empire's Chancellor from 1871 to 1890. Many of these events occurred after Riemann's death, but as we shall see, he was affected by the war of 1866. In this war, Prussia (which had proposed a radical and unwelcome reshaping of the German confederation) faced Austria, which allied with most of the "third Germany" (states other than Prussia and Austria). Prussia won the battle of Königgratz (July 1866) and then annexed much north German territory, including the Kingdom of Hanover.

Returning to March 1848, Riemann was loyal to the establishment and, as a member of the student corps, was on guard duty at the Royal Palace from 9 a.m. on 24 March to 1 p.m. the next day. This seems to have been the only time he took sides in political matters.

At Easter 1849 Riemann returned to Göttingen and attended courses of lectures on philosophy and experimental physics. Here his teacher, Wilhelm Weber (1804–1891) was a remarkable man who also had much influence on Riemann. Weber became a professor of physics in Göttingen in 1831 at the instigation of Gauss. Gauss and Weber published important joint work on magnetism. Their famous telegraph, the first effective working one, dated from 1833 and permitted simultaneous magnetic observations at the Physical Laboratory and the Astronomical Observatory two miles away. There was an interruption in Weber's work in 1837 when, with six other professors, he was dismissed for a protest concerning the revocation of the Hanoverian constitution of 1833. After traveling, and working at Leipzig, he was reinstated in 1849. His later work focused on electrodynamics. In 1850 a seminar in mathematical physics was formed, led by Weber, in which Riemann joined. To us it is remarkable that he could be heavily involved in experimental work at the same time he was preparing his groundbreaking doctoral thesis, "Foundations for a general theory of functions of a complex variable." Riemann published papers in physics throughout his short life. His paper [23] seems to be the first in which it is asserted that electromagnetic effects propagate at the speed of light. In his paper [19] on propagation of waves in a compressible gas, the concept of a shock wave appears for the first time.

Of incomparably greater importance for Riemann's legacy is the doctoral thesis [14]. (It seems that Gauss, the formal thesis adviser, gave Riemann absolutely no help. Gauss's remark in the thesis interview, that he had long been contemplating

a publication on a related topic, must surely have given Riemann a fright!) The most important matters in the thesis going beyond Cauchy's work are the idea of a Riemann surface, the treatment of complex functions as mappings, with a conformal property, the use of the "Dirichlet principle" (a minimization technique for existence theorems), and the famous "Riemann mapping theorem." Not everything was rigorous, and much later work by others stems from trying to inject the desired rigor into the results. Riemann used the above ideas in several important papers, as we shall see below, quite apart from his application of complex analysis to the zeta function.

Another hurdle remained before Riemann could teach in the university as a Privatdozent. This very junior position was, disagreeably, not salaried and the only income came from the fees paid by students who attended the Privatdozent's lectures. There was no option, however, but to begin this way if Riemann wanted to be a university teacher. The obstacle that remained was the habilitation thesis. The topic, the theory of trigonometric series, was suggested by Dirichlet, who spent some time in Göttingen in the autumn of 1852. Dirichlet provided Riemann with references to previous work. Such references were dreadfully lacking in Riemann's doctoral thesis; this was not apparently a grievous omission, as it would be today. Dirichlet and Riemann held some of their discussions at Weber's house. In December 1853 Riemann turned in his thesis "On the representation of a function by a trigonometric series" to the dean, but because Gauss was in very poor health, it was not until Whitsun 1854 that the examination and trial lecture took place. The trial lecture was to become a landmark in the history of geometry, but we must first of all look at the habilitation thesis [21], which was also an important contribution to pure mathematics. The thesis was published posthumously by Dedekind from a complete manuscript in the "Nachlass," the large body of miscellaneous writings left a Riemann's death, and still available for study at the university library in Göttingen.

Riemann was concerned in the thesis with necessary conditions for the representation of a function by a trigonometric series. His idea of using the function F obtained by two integrations of the trigonometric series was new and remains significant: see Zygmund [27, Chap. 9]. Riemann found it necessary to give a rigorous account of integration on the real line which was far superior to earlier work, in the thesis. Indeed, the Riemann integral is standard fare for undergraduates today. Other highlights included the "Riemann-Lebesgue lemma," i.e., the vanishing at infinity of the Fourier coefficients of an integrable function; and the earliest use (it seems) of the principle of stationary phase in §13. The thesis was especially significant for its influence on Georg Cantor's work, but that takes us well beyond Riemann's lifetime.

The "trial lecture" for Riemann's habilitation, a colloquium of the philosophical faculty, was held on June 10, 1854. The title was "On the hypotheses on which geometry is based." Riemann described a general mathematical object, an n-dimensional differentiable manifold. The metric was based on a smoothly varying inner product on the tangent space (a "Riemannian metric"). He indicated that this was conceivably an accurate description of physical space (rather than the simplest case, Euclidean space), an idea the audience would have automatically detested. He

went to considerable lengths to avoid formulas (there is only one in the published form of the whole paper). After being published from the Nachlass in 1868 [22], the lecture stimulated differential geometers for the rest of the century. Indeed, deep new results in Riemannian geometry continue to appear, such as Gromov's "almost flat manifold theorem" of 1978 [8]. There is also a certain relation to the mathematical basis of the theory of general relativity, published by Einstein in 1915, which surely would have delighted Riemann.

Readers might expect that after obtaining powerful new results in each of his two theses and the trial lecture, Riemann would have been invited to speak at conferences and seminars. This is to misunderstand the nature of nineteenth century mathematics research. Göttingen, one of the best universities in the world in the 1850s, had only one annual scientific meeting, which covered the natural sciences and medicine. Colloquium lectures by visiting scientists were virtually unknown. In mathematics, Felix Klein and Georg Cantor put forward the idea of an international congress only in the 1890s, the first one taking place in Zurich in 1897. Of course, there was some travel by individual mathematicians to meet colleagues in other states and countries; we shall see some more of this as the story continues.

Richard Dedekind (1831–1916), Gauss's last student, completed his habilitation just a few weeks after Riemann. At this stage, his work was at a rather pedestrian level, but it was not too long before his extraordinary talents came to the fore. He gave the first rigorous construction of the real numbers in 1858 using "Dedekind cuts," although this was not published until 1872. His work on algebraic number fields is well known and of major importance, indeed its influence extends to abstract algebra as a whole. He was the first to prove the "Schröder-Bernstein theorem" of 1897, on comparability of infinite cardinal numbers. Dedekind obtained the result in 1887. It is remarkable that he was reluctant to publish such outstanding results. Going back to the 1850s, the two young instructors became friends, and the sociable and cheerful Dedekind was invaluable in lifting Riemann's often plummeting spirits. We have records of the early courses taught by both scholars, and it seems that Galois theory was taught in a university for the first time by Dedekind in winter 1856/1857. Riemann, meanwhile, lectured regularly on his new complex function theory, among other topics.

Riemann's stock rose considerably when he published "Contributions to the theory of functions represented by the Gauss series $F(\alpha, \beta, \gamma, x)$" in the proceedings of the Göttingen Academy in 1857 [15]. He provided an entirely new approach to the hypergeometric function, studied by Gauss [7] and Kummer [10]. He specified the P-functions geometrically and used (in effect) groups of matrices to specify the behavior of P-functions under circuits of the branch points. All the results obtained via Kummer's laborious derivations, much admired at the time, are obtained "almost without calculation" as Riemann remarks in an announcement of the paper [16]. The paper influenced later work of Fuchs and Hilbert, among others.

Even more important for Riemann's reputation was his treatment of a hot topic in "The theory of Abelian functions" in Borchardt's Journal, 1857 [17]. The underlying idea is the study of the integrals of a rational function

$$\int_0^z R(x, y)dx$$

where y is an algebraic function, say $F(x, y) = 0$. Jacobi had shown in 1834 that if F is nonsingular of degree greater than 3, there will be more than two periods to such an integral. Weierstrass had treated $F(x, y) = y^2 - f(x)$, $\deg f = 2n + 1$, in papers of 1853 and 1856. Riemann used entirely different methods, requiring Riemann surfaces and the Dirichlet principle. See Weyl [26] for a rigorous account of Riemann's results, an assessment of which could take us much too far afield here. In particular, there are key concepts in topology and algebraic geometry that derive their lineage from this crucial paper. The effort he put into it sent him into a state of mental exhaustion, and Dedekind describes in [4] how a walking trip in the Harz mountains that the two of them undertook restored Riemann's spirits. It seems that they discussed Newton rather than pure mathematics. Dedekind deserves the gratitude of posterity for his efforts.

On 23 February 1855, Gauss had died and Dirichlet had succeeded him in Göttingen. Riemann's father and his sister Clara also died in this year. It was just as well that Riemann was given at this time a small stipend of 200 thalers a year from the government, since he had never been able to avoid accepting financial help from his father. Arguably the three best mathematicians in the world were together in Göttingen in the mid-1850s, but Dirichlet's health gave way before too long and Dedekind was to leave for a professorship at the Polytechnic school in Zürich in 1858. (He spent the whole of his career from 1862 in his native Braunschweig.) In 1857, Riemann was promoted to extraordinary professor and his stipend increased to 300 thalers a year. But then his spirits were crushed by the death of his beloved brother Wilhelm, a clerk in the post office. Riemann now needed to provide for the upkeep of his three remaining sisters. (At this time the doors of the universities and professions were firmly closed to women, and the millions of women who worked only had access to menial jobs such as farm work, domestic service, and sweatshops [1].) After the sisters moved to Göttingen, the youngest, Marie, died in the spring of 1858. Nevertheless, Riemann gradually became happier and more self-confident living with family; he had always made trips back to see his family whenever he could.

In the autumn of 1858, Riemann got to know the Italian mathematicians Francesco Brioschi (1824–1897), Enrico Betti (1823–1892), and Felice Casorati (1835–1890), who were on a tour that took in Paris, Berlin, and Göttingen. These men were hoping to build up Italian mathematics and intended to gain from their contacts with the best foreign mathematicians. Along with Eugenio Beltrami (1835–1900), a student of Brioschi, they all played a significant part in the rise of Italian mathematics in the following years. This was also the starting point of Riemann's contacts with Italy, which were so valuable to him in his last years.

Dirichlet died in May 1859 after a long period of heart problems. Riemann was now deservedly appointed as his replacement and given living quarters at the Observatory. (The Göttingen Observatory retains its original appearance and since 2009 has housed the Lichtenberg-Kolleg Institute for advanced study in the humanities and social sciences.) Riemann was only thirty-two, and with good luck could have now looked forward to a prosperous and productive future beyond the end of the century. Sadly it was not to be. The reader may be astonished at the short

lives of many of the characters in our narrative. It is well to note that for a man born in Germany in the 1870s, life expectancy was 36.5; for a woman, 38.5 [1]. Riemann's lifespan of almost 40 years was not too bad when looked at in this way.

In September 1859, Riemann visited the Berlin Academy of Sciences, which had recently elected him a corresponding member. Dedekind traveled with him, and he enjoyed a friendly welcome from the leading mathematicians Eduard Kummer (1810–1893), Carl Borchardt (1817–1880), Leopold Kronecker (1823–1891), and Karl Weierstrass (1815–1897). All of these will be familiar to the reader except perhaps Borchardt, who worked on the arithmetic–geometric mean problem (begun by Gauss) and continued Kummer's work on the secular disturbances of the planets. In total he wrote 25 papers, as well as editing Crelle's Journal from 1856 to 1880.

Riemann's visit was followed by the submission to the Berlin Academy of his paper [18] on zeta function and the frequency of prime numbers in October 1859. We defer discussion of this splendid paper until the end of our account of Riemann's life.

Like the Italian mathematicians, Riemann also visited Paris, spending a month there in 1860. This time he was able to meet the excellent scholars Joseph Serret (1819–1885), Joseph Bertrand (1822–1900), Charles Hermite (1822–1901), Victor Puiseux (1820–1883), Charles Briot (1817–1882), and Jean Claude Bouquet (1819–1885). Of these, Joseph Bertrand may be the least familiar. He worked on analytical mechanics, thermodynamics, statistical probability, and the theory of curves and surfaces, as well as editing *Journal des Savants* (1865–1900). He also created "Bertrand's postulate": for any $n > 3$ there is a prime number p in $(n, 2n - 2)$. This was proved by Chebyshev in 1852.

Riemann was finally well enough off to contemplate marriage, and in June 1862 he married Elise Koch, a friend of his sisters. Now his health began to deteriorate, and she was his great support through his remaining 4 years. In July 1862 he had an attack of pleurisy, and this seems to have led to the lung disease that shortened his life. On the advice of Riemann's doctor, Weber and the geology professor Sartorius von Waltershausen (1809–1876) pressed the authorities to give Riemann leave and financial support for a visit to Italy that would benefit his health. Riemann and Elise left in November 1862 and initially stayed with the consul in Messina, who was a friend of Sartorius. The long visit to Italy allowed him to visit many of the most famous cities and enjoy the art treasures and antiquities, as well as spending time in discussions with Enrico Betti in Pisa. Dedekind believed that Riemann enjoyed the greatest happiness of his life on his visits to Italy, partly because the people he encountered were so relaxed and informal.

Not long after his return, his health was so bad that he began a second voyage to Italy in August 1863. When they were in Pisa, Elise gave birth to their daughter Ida. In May 1864 he took a villa in Pisa, and here at the end of August, his younger sister Helene died. He was offered an appointment at the University of Pisa, but felt his health was too precarious to accept. The next winter was passed in Pisa with his learned friends Betti, Beltrami, and others. He was now working on the memoir on the vanishing of theta functions [20]. In spite of very poor health, he went back to Göttingen, hoping to resume his teaching, arriving on 3 October 1865. His last

scientific work was a memoir on the mechanism of the ear, which he was unable to finish. On 15 June 1866, he and Elise set off to Italy. Their journey was interrupted at Cassel, where the war mentioned above had caused the destruction of the railway; but they did manage to complete the journey to Lake Maggiore, where he worked quietly, and enjoyed the scenery, until his death on 20 July, with Elise at his side.

For further reading about Riemann's life, see the book of Laugwitz [11] and the references therein.

2 The Zeta Function

We now turn to Riemann's memoir on the zeta function. Euler had given many results in number theory using product formulae; one of these was the formula for

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad (s \in (1, \infty)),$$

namely

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1} \quad (s > 1).$$

He used this to show that $\sum_{p} p^{-1}$ diverges. Riemann viewed $\zeta(s)$ as a function of a complex variable and used contour integration to show that

$$\pi^{-\frac{1}{2}s}\, \Gamma\left(\frac{1}{2}s\right) \zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} \left(x^{\frac{1}{2}s-1} + x^{-\frac{1}{2}s-\frac{1}{2}}\right) \omega(x) dx \tag{1}$$

where

$$\omega(x) = \sum_{1}^{\infty} e^{-n^2 \pi x}$$

is related to a Jacobi theta function. Since the right side is unchanged if we replace s by $1 - s$, this gives both the analytic continuation of $\zeta(s)$ over the whole plane and the functional equation for $\zeta(s)$. We also easily obtain the set of zeros of $\zeta(s)$ in $\operatorname{Re} s < 0$, namely $\{-2, -4, -6, \ldots\}$. Riemann gives a second proof of the functional equation. His next four assertions about $\zeta(s)$ are backed up by a number of indications of proof; Davenport [3] describes them as conjectures, but Riemann (as we know from his correspondence with Weierstrass) thought that just some details needed filling in.

1. $\zeta(s)$ has infinitely many zeros in the critical strip. (These must clearly be placed symmetrically with respect to the real axis and with respect to the line $\operatorname{Re} s = \frac{1}{2}$.)
2. The number $N(T)$ of zeros of $\zeta(s)$ in the "critical strip," $0 \le \operatorname{Re} s \le 1$ with $0 < t \le T$ satisfies

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

   It was not until 1905 that von Mangoldt gave the full proof of this (he had the result, with a worse error term, by 1895).
3. The entire function $\xi(s)$ defined by

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right) \zeta(s)$$

   has the product representation

$$\xi(s) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho},$$

   where A and B are constants. Here and below, ρ runs through the zeros of $\zeta(s)$ in the critical strip. In order to get a rigorous proof, Jacques Hadamard (1865–1963), in the 1890s, developed the theory of entire functions of finite order, the distribution of their zeros, and their product representations. Hadamard's proof of (3) dates to 1893.
4. We can write out an explicit formula for $\pi(x) - \operatorname{li} x$ in terms of the zeros of $\zeta(s)$. This looks a lot tidier in the form proved by von Mangoldt in 1895. Let $\Lambda(n)$ denote $\log p$ if n is a prime power p^k, and $\Lambda(n) = 0$ otherwise. Then, for $x \notin \mathbb{Z}$,

$$\sum_{n \le x} \Lambda(n) - x = -\sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1 - x^{-2}),$$

   where we must group ρ with $\bar{\rho}$ in the sum on the right. Riemann's sketch proof involves an application of Fourier inversion but omits some key details.

Riemann conjectures early in the paper that all complex zeros of $\zeta(s)$ lie on the "critical line" $\operatorname{Re} s = \frac{1}{2}$. He says that after a few brief and fruitless attempts to prove this, he has put it on one side for the time being, because it did not seem to be essential to the immediate object of his investigation. Of course the Riemann hypothesis is now one of the best known unsolved problems in all of mathematics. We do know that more than 41% of the zeros of the zeta function up to a large height T are on the critical line [2]. Riemann makes the assertion [18] that he can prove this with asymptotically 100% in place of 41%; this still seems out of reach.

It certainly seems on reading Riemann's memoir that he regards the asymptotic formula (conjectured by Gauss)

$$\pi(x) \sim \operatorname{li} x$$

as a consequence of his investigation, but this theorem was only proved rigorously in 1895, by both Hadamard and Charles de la Vallée Poussin (1866–1962). A key element of the proof of this theorem, the "prime number theorem," is the demonstration that no zero of $\zeta(s)$ has real part 1. Davenport [3] is the best source for the proofs of the results described above.

There is a remarkable postscript to Riemann's published work on the zeta function. In 1932 Carl Ludwig Siegel [24] published an account of material about the zeta function that he had found in the Nachlass. Bear in mind that Dedekind and Heinrich Weber had already combed through the Nachlass to find anything that could be included, as a coherent result, in their two editions of Riemann's collected works! There are two topics covered in Siegel's paper. One is an asymptotic formula for the computation of the real function

$$Z(t) = e^{i\theta(t)}\zeta\left(\frac{1}{2} + it\right)$$

where $\theta(t)$ is defined so that the sign of $\zeta\left(\frac{1}{2} + it\right)$ is opposite to that of $Z(t)$. The other topic is a new representation of $\zeta(s)$ in terms of definite integrals. Riemann attained the formula

$$Z(t) = \sum_{n=1}^{N} n^{-\frac{1}{2}} 2\cos(\theta(t) - t\log n) + \frac{e^{-i\theta(t)+i\pi/4}e^{-t\pi/2}}{(2\pi)^{1/2+it}(1 - ie^{-t\pi})}\int_{C_N} \frac{(-x)^{\frac{1}{2}+it}e^{-Nx}dx}{e^x - 1}$$

where C_N is a certain path which begins and ends on the real axis at $+\infty$. This is amenable to the saddle point method; see Edwards [6, Chap. 9] for a discussion. As for the definite integral representation of $\zeta(s)$ mentioned above, this is said by Siegel to have been found in the Nachlass in 1926 by Erich Bessel-Hagen (1898–1946). Bessel-Hagen, a friend of Siegel, was then working on his habilitation at Göttingen. See [6, Chap. 7] for more details. We can certainly say that Riemann had obtained privately far more results about the zeta function than can be found in his memoir for the Berlin Academy, and that these led him to accurate empirical discoveries (such as the location of the first few zeros in the critical strip) that made his hypothesis on the zeta function plausible.

For further study of the zeta function, besides the books of Davenport and Edwards mentioned above, there are interesting and well-written books by Titchmarsh [25], Ivic [9], and Patterson [13].

In summary, we can say that Riemann had tremendous insights in every branch of mathematics that he studied in any depth, and that he is quite an awe-inspiring figure. It is certainly still worthwhile to read his papers and to try and picture how his "fantastic premonitions" (as Dedekind describes them) looked to his contemporaries.

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Ramanujan's Formula for $\zeta(2n+1)$

Bruce C. Berndt and Armin Straub

Abstract Ramanujan made many beautiful and elegant discoveries in his short life of 32 years, and one of them that has attracted the attention of several mathematicians over the years is his intriguing formula for $\zeta(2n+1)$. To be sure, Ramanujan's formula does not possess the elegance of Euler's formula for $\zeta(2n)$, nor does it provide direct arithmetical information. But, one of the goals of this survey is to convince readers that it is indeed a remarkable formula. In particular, we discuss the history of Ramanujan's formula, its connection to modular forms, as well as the remarkable properties of the associated polynomials. We also indicate analogues, generalizations and opportunities for further research.

1 Introduction

As customary, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, $\operatorname{Re} s > 1$, denotes the Riemann zeta function. Let B_r, $r \geq 0$, denote the r-th Bernoulli number. When n is a positive integer, Euler's formula

$$\zeta(2n) = \frac{(-1)^{n-1} B_{2n}}{2(2n)!} (2\pi)^{2n} \tag{1}$$

not only provides an elegant formula for evaluating $\zeta(2n)$, but it also tells us of the arithmetical nature of $\zeta(2n)$. In contrast, we know very little about the odd zeta values $\zeta(2n+1)$. One of the major achievements in number theory in the past half-century is R. Apéry's proof that $\zeta(3)$ is irrational [2], but for $n \geq 2$, the arithmetical nature of $\zeta(2n+1)$ remains open.

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H. Montgomery et al. (eds.), *Exploring the Riemann Zeta Function*,
DOI 10.1007/978-3-319-59969-4_2

Ramanujan made many beautiful and elegant discoveries in his short life of 32 years, and one of them that has attracted the attention of several mathematicians over the years is his intriguing formula for $\zeta(2n+1)$. To be sure, Ramanujan's formula does not possess the elegance of (1), nor does it provide any arithmetical information. But, one of the goals of this survey is to convince readers that it is indeed a remarkable formula.

Theorem 1.1 (Ramanujan's Formula for $\zeta(2n+1)$) *Let B_r, $r \geq 0$, denote the r-th Bernoulli number. If α and β are positive numbers such that $\alpha\beta = \pi^2$, and if n is a positive integer, then*

$$\alpha^{-n}\left(\frac{1}{2}\zeta(2n+1)+\sum_{m=1}^{\infty}\frac{1}{m^{2n+1}(e^{2m\alpha}-1)}\right)$$
$$-(-\beta)^{-n}\left(\frac{1}{2}\zeta(2n+1)+\sum_{m=1}^{\infty}\frac{1}{m^{2n+1}(e^{2m\beta}-1)}\right)$$
$$=2^{2n}\sum_{k=0}^{n+1}(-1)^{k-1}\frac{B_{2k}}{(2k)!}\frac{B_{2n+2-2k}}{(2n+2-2k)!}\alpha^{n+1-k}\beta^{k}. \tag{2}$$

Theorem 1.1 appears as Entry 21(i) in Chap. 14 of Ramanujan's second notebook [58, p. 173]. It also appears in a formerly unpublished manuscript of Ramanujan that was published in its original handwritten form with his lost notebook [59, formula (28), pp. 319–320].

The purposes of this paper are to convince readers why (2) is a fascinating formula, to discuss the history of (2) and formulas surrounding it, and to discuss the remarkable properties of the polynomials on the right-hand side of (2). We briefly point the readers to analogues and generalizations at the end of our paper.

In Sect. 2, we discuss Ramanujan's aforementioned unpublished manuscript and his faulty argument in attempting to prove (2). Companion formulas (both correct and incorrect) with the same parentage are also examined. In the following Sect. 3, we offer an alternative formulation of (2) in terms of hyperbolic cotangent sums. In Sects. 4 and 5, we then discuss a more modern interpretation of Ramanujan's identity. We introduce Eisenstein series and their Eichler integrals, and observe that their transformation properties are encoded in (2). In particular, this leads us to a vast extension of Ramanujan's identity from Eisenstein series to general modular forms.

In a different direction, (2) is a special case of a general transformation formula for analytic Eisenstein series, or, in another context, a general transformation formula that greatly generalizes the transformation formula for the logarithm of the Dedekind eta function. We show that Euler's famous formula for $\zeta(2n)$ arises from the same general transformation formula, and so Ramanujan's formula (2) is a natural analogue of Euler's formula. All of this is discussed in Sect. 6.

In Sect. 7, we discuss some of the remarkable properties of the polynomials that appear in (2). These polynomials have received considerable recent attention, with exciting extensions by various authors to other modular forms. We sketch recent developments and indicate opportunities for further research. We then provide in Sect. 8 a brief compendium of proofs of (2), or its equivalent formulation with hyperbolic cotangent sums.

2 Ramanujan's Unpublished Manuscript

The aforementioned handwritten manuscript containing Ramanujan's formula for $\zeta(2n+1)$ was published for the first time with Ramanujan's lost notebook [59, pp. 318–321]. This partial manuscript was initially examined in detail by the first author in [12], and by Andrews and the first author in their fourth book on Ramanujan's lost notebook [1, Chap. 12, pp. 265–284].

The manuscript's content strongly suggests that it was intended to be a continuation of Ramanujan's paper [55], [57, pp. 133–135]. It begins with paragraph numbered 18, giving further evidence that it was intended to be a part of [55]. One might therefore ask why Ramanujan did not incorporate this partial manuscript in his paper [55]. As we shall see, one of the primary claims in his manuscript is false. Ramanujan's incorrect proof arose from a partial fraction decomposition, but because he did not have a firm grasp of the Mittag–Leffler Theorem, he was unable to discern his mis-application of the theorem. Most certainly, Ramanujan was fully aware that he had indeed made a mistake, and consequently he wisely chose not to incorporate the results in this manuscript in his paper [55]. Ramanujan's mistake arose when he attempted to find the partial fraction expansion of $\cot(\sqrt{w\alpha})\coth(\sqrt{w\beta})$. We now offer Ramanujan's incorrect assertion from his formerly unpublished manuscript.

Entry 2.1 (p. 318, Formula (21)) *If α and β are positive numbers such that $\alpha\beta = \pi^2$, then*

$$\frac{1}{2w} + \sum_{m=1}^{\infty}\left\{\frac{m\alpha\coth(m\alpha)}{w+m^2\alpha} + \frac{m\beta\coth(m\beta)}{w-m^2\beta}\right\} = \frac{\pi}{2}\cot(\sqrt{w\alpha})\coth(\sqrt{w\beta}). \tag{3}$$

We do not know if Ramanujan was aware of the Mittag–Leffler Theorem. Normally, *if* we could apply this theorem to $\cot(\sqrt{w\alpha})\coth(\sqrt{w\beta})$, we would let $w \to \infty$ and conclude that the difference between the right- and left-hand sides of (3) is an entire function that is identically equal to 0. Of course, in this instance, we cannot make such an argument. We now offer a corrected version of Entry 2.1.

Entry 2.2 (Corrected Version of (3)) *Under the hypotheses of Entry 2.1,*

$$\frac{\pi}{2}\cot(\sqrt{w\alpha})\coth(\sqrt{w\beta}) = \frac{1}{2w} + \frac{1}{2}\log\frac{\beta}{\alpha} + \sum_{m=1}^{\infty}\left\{\frac{m\alpha\coth(m\alpha)}{w+m^2\alpha} + \frac{m\beta\coth(m\beta)}{w-m^2\beta}\right\}. \tag{4}$$

Shortly after stating Entry 2.1, Ramanujan offers the following corollary.

Entry 2.3 (p. 318, Formula (23)) *If α and β are positive numbers such that $\alpha\beta = \pi^2$, then*

$$\alpha\sum_{m=1}^{\infty}\frac{m}{e^{2m\alpha}-1} + \beta\sum_{m=1}^{\infty}\frac{m}{e^{2m\beta}-1} = \frac{\alpha+\beta}{24} - \frac{1}{4}. \tag{5}$$

To prove (5), it is natural to equate coefficients of $1/w$ on both sides of (3). When we do so, we obtain (5), but without the term $-\frac{1}{4}$ on the right-hand side of (5) [1, pp. 274–275]. Ramanujan surely must have been perplexed by this, for he had previously proved (5) by other means. In particular, Ramanujan offered Entry 2.3 as Corollary (i) in Sect. 8 of Chap. 14 in his second notebook [58], [10, p. 255].

Similarly, if one equates constant terms on both sides of (4), we obtain the familiar transformation formula for the logarithm of the Dedekind eta function.

Entry 2.4 (p. 320, Formula (29)) *If α and β are positive numbers such that $\alpha\beta = \pi^2$, then*

$$\sum_{m=1}^{\infty}\frac{1}{m(e^{2m\alpha}-1)} - \sum_{m=1}^{\infty}\frac{1}{m(e^{2m\beta}-1)} = \frac{1}{4}\log\frac{\alpha}{\beta} - \frac{\alpha-\beta}{12}. \tag{6}$$

Of course, if we had employed (3) instead of (4), we would not have obtained the expression $\frac{1}{4}\log\frac{\alpha}{\beta}$ on the right-hand side of (6). Entry 2.4 is stated by Ramanujan as Corollary (ii) in Sect. 8 of Chap. 14 in his second notebook [58], [10, p. 256] and as Entry 27(iii) in Chap. 16 of his second notebook [58], [11, p. 43].

In contrast to Entries 2.3 and 2.4, Ramanujan's formula (2) can be derived from (3), because we equate coefficients of w^n, $n \geq 1$, on both sides, and so the missing terms in (3) do not come into consideration. We now give the argument likely given by Ramanujan. (Our exposition is taken from [1, p. 278].)

Proof Return to (3), use the equality

$$\coth x = 1 + \frac{2}{e^{2x}-1}, \tag{7}$$

and expand the summands into geometric series to arrive at

$$\frac{\pi}{2}\cot(\sqrt{w\alpha})\coth(\sqrt{w\beta}) = \frac{1}{2w} + \sum_{m=1}^{\infty}\left\{\frac{1}{m}\sum_{k=0}^{\infty}\left(-\frac{w}{m^2\alpha}\right)^k\left(1+\frac{2}{e^{2m\alpha}-1}\right)\right.$$
$$\left. -\frac{1}{m}\sum_{k=0}^{\infty}\left(\frac{w}{m^2\beta}\right)^k\left(1+\frac{2}{e^{2m\beta}-1}\right)\right\}. \tag{8}$$

Following Ramanujan, we equate coefficients of w^n, $n \geq 1$, on both sides of (8). On the right side, the coefficient of w^n equals

$$(-\alpha)^{-n}\zeta(2n+1) + 2(-\alpha)^{-n}\sum_{m=1}^{\infty}\frac{1}{m^{2n+1}(e^{2m\alpha}-1)}$$
$$-\beta^{-n}\zeta(2n+1) + 2\beta^{-n}\sum_{m=1}^{\infty}\frac{1}{m^{2n+1}(e^{2m\beta}-1)}. \tag{9}$$

Using the Laurent expansions for $\cot z$ and $\coth z$ about $z = 0$, we find that on the left side of (8),

$$\frac{\pi}{2}\cot(\sqrt{w\alpha})\coth(\sqrt{w\beta}) = \frac{\pi}{2}\sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k}B_{2k}}{(2k)!}(w\alpha)^{k-1/2}\sum_{j=0}^{\infty}\frac{2^{2j}B_{2j}}{(2j)!}(w\beta)^{j-1/2}. \tag{10}$$

The coefficient of w^n in (10) is easily seen to be equal to

$$2^{2n+1}\sum_{k=0}^{n+1}(-1)^k\frac{B_{2k}}{(2k)!}\frac{B_{2n+2-2k}}{(2n+2-2k)!}\alpha^k\beta^{n+1-k}, \tag{11}$$

where we used the equality $\alpha\beta = \pi^2$. Now equate the expressions in (9) and (11), then multiply both sides by $(-1)^n\frac{1}{2}$, and lastly replace k by $n+1-k$ in the finite sum. The identity (2) immediately follows. □

As mentioned in Sect. 1, Ramanujan also recorded (2) as Entry 21(i) of Chap. 14 in his second notebook [58, p. 173], [10, p. 271]. Prior to that on page 171, he offered the partial fraction decomposition

$$\pi^2 xy\cot(\pi x)\coth(\pi y) = 1 + 2\pi xy\sum_{n=1}^{\infty}\frac{n\coth(\pi nx/y)}{n^2+y^2} - 2\pi xy\sum_{n=1}^{\infty}\frac{n\coth(\pi ny/x)}{n^2-x^2}, \tag{12}$$

which should be compared with (3). The two infinite series on the right side of (12) diverge individually, but when combined together into one series, the series converges. Thus, most likely, Ramanujan's derivation of (2) when he recorded Entry 21(i) in his second notebook was similar to the argument that he used in his partial manuscript.

Sitaramachandrarao [69] modified Ramanujan's approach via partial fractions to avoid the pitfalls experienced by Ramanujan. Sitaramachandrarao established the partial fraction decomposition

$$\pi^2 xy\cot(\pi x)\coth(\pi y) = 1 + \frac{\pi^2}{3}(y^2 - x^2) - 2\pi xy\sum_{m=1}^{\infty}\left(\frac{y^2\coth(\pi mx/y)}{m(m^2+y^2)} + \frac{x^2\coth(\pi my/x)}{m(m^2-x^2)}\right). \tag{13}$$

Using the elementary identities

$$\frac{y^2}{m(m^2+y^2)} = -\frac{m}{m^2+y^2} + \frac{1}{m}$$

and

$$\frac{x^2}{m(m^2-x^2)} = \frac{m}{m^2-x^2} - \frac{1}{m},$$

and then employing (7), he showed that

$$\pi^2 xy\cot(\pi x)\coth(\pi y) = 1 + \frac{\pi^2}{3}(y^2 - x^2) + 2\pi xy\sum_{m=1}^{\infty}\left(\frac{m\coth(\pi mx/y)}{m^2+y^2} - \frac{m\coth(\pi my/x)}{m^2-x^2}\right) - 4\pi xy\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{1}{e^{2\pi mx/y}-1} - \frac{1}{e^{2\pi my/x}-1}\right). \tag{14}$$

Setting $\pi x = \sqrt{w\alpha}$ and $\pi y = \sqrt{w\beta}$ above and invoking the transformation formula for the logarithm of the Dedekind eta function from Entry 2.4, we can readily deduce (4). For more details, see [1, pp. 272–273].

The first published proof of (2) is due to Malurkar [49] in 1925–1926. Almost certainly, he was unaware that (2) can be found in Ramanujan's notebooks [58].

If we set $\alpha = \beta = \pi$ and replace n by $2n+1$ in (2), we deduce that, for $n \geq 0$,

$$\zeta(4n+3) = 2^{4n+2}\pi^{4n+3}\sum_{k=0}^{2n+2}(-1)^{k+1}\frac{B_{2k}}{(2k)!}\frac{B_{4n+4-2k}}{(4n+4-2k)!} - 2\sum_{k=1}^{\infty}\frac{k^{-4n-3}}{e^{2\pi k}-1}. \tag{15}$$

This special case is actually due to Lerch [44] in 1901. The identity (15) is a remarkable identity, for it shows that $\zeta(4n+3)$ is equal to a rational multiple of π^{4n+3} plus a very rapidly convergent series. Therefore, $\zeta(4n+3)$ is "almost" a rational multiple of π^{4n+3}.

3 An Alternative Formulation of (2) in Terms of Hyperbolic Cotangent Sums

If we use the elementary identity (7), we can transform Ramanujan's identify (2) for $\zeta(2n+1)$ into an identity for hyperbolic cotangent sums, namely,

$$\alpha^{-n}\sum_{m=1}^{\infty}\frac{\coth(\alpha m)}{m^{2n+1}} = (-\beta)^{-n}\sum_{m=1}^{\infty}\frac{\coth(\beta m)}{m^{2n+1}} - 2^{2n+1}\sum_{k=0}^{n+1}(-1)^k\frac{B_{2k}}{(2k)!}\frac{B_{2n+2-2k}}{(2n+2-2k)!}\alpha^{n+1-k}\beta^k, \tag{16}$$

where, as before, n is a positive integer and $\alpha\beta = \pi^2$. To the best of our knowledge, the first recorded proof of Ramanujan's formula (2) in the form (16) was by Nanjundiah [52] in 1951. If we replace n by $2n+1$ and set $\alpha = \beta = \pi$, then (16) reduces to

$$\sum_{m=1}^{\infty}\frac{\coth(\pi m)}{m^{4n+3}} = 2^{4n+2}\pi^{4n+3}\sum_{k=0}^{2n+2}(-1)^{k+1}\frac{B_{2k}}{(2k)!}\frac{B_{4n+4-2k}}{(4n+4-2k)!}. \tag{17}$$

This variation of (15) was also first established by Lerch [44]. Later proofs were given by Watson [72], Sandham [64], Smart [70], Sayer [65], Sitaramachandrarao [69], and the first author [7, 8]. The special cases $n = 0$ and $n = 1$ are Entries 25(i), (ii) in Chap. 14 of Ramanujan's second notebook [58, p. 176], [10, p. 293]. He communicated this second special case in his first letter to Hardy [57, p. xxvi].

Significantly generalizing an idea of Siegel in deriving the transformation formula for the Dedekind eta function [68], Kongsiriwong [41] not only established (16), but he derived several beautiful generalizations and analogues of (16).

Deriving a very general transformation formula for Barnes' multiple zeta function, Komori, Matsumoto, and Tsumura [40] established not only (16), but also a plethora of further identities and summations in closed form for series involving hyperbolic trigonometric functions.

4 Eisenstein Series

A more modern interpretation of Ramanujan's remarkable formula, which is discussed, for instance, in [30], is that (2) encodes the fundamental transformation properties of Eisenstein series of level 1 and their Eichler integrals. The goal of this section and the next is to explain this connection and to introduce these modular objects starting with Ramanujan's formula. As in [26] and [30], set

$$F_a(z) = \sum_{n=1}^{\infty} \sigma_{-a}(n) e^{2\pi i n z}, \quad \sigma_k(n) = \sum_{d|n} d^k. \tag{18}$$

Observe that, with $q = e^{2\pi i z}$, we can express $F_a(z)$ as a Lambert series

$$F_a(z) = \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{-a} \right) q^n = \sum_{d=1}^{\infty} \sum_{m=1}^{\infty} d^{-a} q^{dm} = \sum_{n=1}^{\infty} \frac{n^{-a} q^n}{1-q^n} = \sum_{n=1}^{\infty} \frac{n^{-a}}{e^{-2\pi i n z} - 1}$$

in the form appearing in Ramanujan's formula (2). Indeed, if we let $z = \alpha i/\pi$, then Ramanujan's formula (2) translates into

$$\left\{ \frac{\zeta(2m+1)}{2} + F_{2m+1}(z) \right\} = z^{2m} \left\{ \frac{\zeta(2m+1)}{2} + F_{2m+1}\left(-\frac{1}{z}\right) \right\} + \frac{(2\pi i)^{2m+1}}{2z} \sum_{n=0}^{m+1} \frac{B_{2n}}{(2n)!} \frac{B_{2m-2n+2}}{(2m-2n+2)!} z^{2n}. \tag{19}$$

This generalization to values z in the upper half-plane $\mathscr{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ was derived by Grosswald in [25].

Ramanujan's formula becomes particularly simple if the integer m satisfies $m < -1$. In that case, with $k = -2m$, Eq. (19) can be written as

$$E_k(z) = z^{-k} E_k(-1/z), \tag{20}$$

with

$$E_k(z) = 1 + \frac{2}{\zeta(1-k)} F_{1-k}(z) = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n. \tag{21}$$

The series E_k, for even $k > 2$, are known as the *normalized Eisenstein series* of weight k. They are fundamental instances of modular forms. A *modular form*

of weight k and level 1 is a function G on the upper half-plane $\mathscr{H}$, which is holomorphic on $\mathscr{H}$ (and as $z \to i\infty$), and transforms according to

$$(cz+d)^{-k}G(Vz) = G(z) \tag{22}$$

for each matrix

$$V = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Here, as usual, $\mathrm{SL}_2(\mathbb{Z})$ is the modular group of integer matrices V with determinant 1, and these act on $\mathscr{H}$ by fractional linear transformations as

$$Vz = \frac{az+b}{cz+d}. \tag{23}$$

Modular forms of higher level are similarly defined and transform only with respect to certain subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Equation (20), together with the trivial identity $E_k(z+1) = E_k(z)$, establishes that $E_k(z)$ satisfies the modular transformation (22) for the matrices

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

It is well known that S and T generate the modular group $\mathrm{SL}_2(\mathbb{Z})$, from which it follows that $E_k(z)$ is invariant under the action of any element in $\mathrm{SL}_2(\mathbb{Z})$. In other words, $E_k(z)$ satisfies (22) for all matrices in $\mathrm{SL}_2(\mathbb{Z})$.

To summarize our discussion so far, the cases $m < -1$ of Ramanujan's formula (2) express the fact that, for even $k > 2$, the q-series $E_k(z)$, defined in (21) as the generating function of sums of powers of divisors, transforms like a modular form of weight k. Similarly, the case $m = -1$ encodes the fact that

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n,$$

as defined in (21), transforms according to

$$E_2(z) = z^{-2}E_2(-1/z) - \frac{6}{\pi i z}.$$

This is an immediate consequence of Ramanujan's formula in the form (19). The weight 2 Eisenstein series $E_2(z)$ is the fundamental instance of a *quasimodular form*.

In the next section, we discuss the case $m > 0$ of Ramanujan's formula (2) and describe its connection with Eichler integrals.

5 Eichler Integrals and Period Polynomials

If f is a modular form of weight k and level 1, then

$$z^{-k}f\left(-\frac{1}{z}\right)-f(z)=0.$$

The derivatives of f, however, do not in general transform in a modular fashion by themselves. For instance, taking the derivative of this modular relation and rearranging, we find that

$$z^{-k-2}f'\left(-\frac{1}{z}\right)-f'(z)=\frac{k}{z}f(z).$$

This is a modular transformation law only if $k=0$, in which case the derivative f' (or, the -1st integral of f) transforms like a modular form of weight 2. Remarkably, it turns out that any $(k-1)$st integral of f does satisfy a nearly modular transformation rule of weight $k-2(k-1)=2-k$. Indeed, if F is a $(k-1)$st primitive of f, then

$$z^{k-2}F\left(-\frac{1}{z}\right)-F(z) \tag{24}$$

is a polynomial of degree at most $k-2$. The function F is called an *Eichler integral* of f and the polynomials are referred to as *period polynomials*.

Let, as usual,

$$D=\frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}z}=q\frac{\mathrm{d}}{\mathrm{d}q}.$$

Remark 5.1 That (24) is a polynomial can be seen as a consequence of *Bol's identity* [16], which states that, for all sufficiently differentiable F and $V=\left[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right]\in \mathrm{SL}_2(\mathbb{R})$,

$$\frac{(D^{k-1}F)(Vz)}{(cz+d)^k}=D^{k-1}[(cz+d)^{k-2}F(Vz)]. \tag{25}$$

Here, Vz is as in (23). If F is an Eichler integral of a modular form $f=D^{k-1}F$ of weight k, then

$$D^{k-1}[(cz+d)^{k-2}F(Vz)-F(z)]=\frac{(D^{k-1}F)(Vz)}{(cz+d)^k}-(D^{k-1}F)(z)=0,$$

for all $V\in \mathrm{SL}_2(\mathbb{Z})$. This shows that $(cz+d)^{k-2}F(Vz)-F(z)$ is a polynomial of degree at most $k-2$. A delightful explanation of Bol's identity (25) in terms of Maass raising operators is given in [45, IV. 2].

Consider, for $m > 0$, the series

$$F_{2m+1}(z) = \sum_{n=1}^{\infty} \sigma_{-2m-1}(n) q^n,$$

which was introduced in (18) and which is featured in Ramanujan's formula (19). Observe that

$$D^{2m+1} F_{2m+1}(z) = \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{-2m-1} \right) n^{2m+1} q^n = \sum_{n=1}^{\infty} \sigma_{2m+1}(n) q^n,$$

and recall from (21) that this is, up to the missing constant term, an Eisenstein series of weight $2m+2$. It thus becomes clear that Ramanujan's formula (19), in the case $m > 0$, encodes the fundamental transformation law of the Eichler integral corresponding to the weight $2m+2$ Eisenstein series.

Remark 5.2 That the right-hand side of (19) is only almost a polynomial is due to the fact that its left-hand side needs to be adjusted for the constant term of the underlying Eisenstein series. To be precise, integrating the weight $k = 2m+2$ Eisenstein series (21), slightly scaled, $k-1$ times, we find that

$$G_{2m+1}(z) := \frac{\zeta(-1-2m)}{2} \frac{(2\pi i z)^{2m+1}}{(2m+1)!} + F_{2m+1}(z) \tag{26}$$

is an associated Eichler integral of weight $-2m$. Keeping in mind the evaluation $\zeta(-1-2m) = -B_{2m+2}/(2m+2)$, Ramanujan's formula in the form (19) therefore can be restated as

$$z^{2m} G_{2m+1}\left(-\frac{1}{z}\right) - G_{2m+1}(z) = \frac{\zeta(2m+1)}{2}(1 - z^{2m}) - \frac{(2\pi i)^{2m+1}}{2} \sum_{n=1}^{m} \frac{B_{2n}}{(2n)!} \frac{B_{2m-2n+2}}{(2m-2n+2)!} z^{2n-1}, \tag{27}$$

where the right-hand side is a polynomial of degree $k-2 = 2m$. Compare, for instance, [74, Eq. (11)].

An interesting and crucial property of period polynomials is that their coefficients encode the critical L-values of the corresponding modular form. Indeed, consider a modular form

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n$$

of weight k. For simplicity, we assume that f is modular on the full modular group $\mathrm{SL}_2(\mathbb{Z})$ (for the case of higher level, we refer to [54]). Let

$$f^*(z) := \frac{a(0)z^{k-1}}{(k-1)!} + \frac{1}{(2\pi i)^{k-1}} \sum_{n=1}^{\infty} \frac{a(n)}{n^{k-1}} q^n$$

be an Eichler integral of f. Then, a special case of a result by Razar [62] and Weil [73] (see also [30, (8)]) shows the following.

Theorem 5.3 *If f and f^* are as described above, then*

$$z^{k-2} f^*\left(-\frac{1}{z}\right) - f^*(z) = -\sum_{j=0}^{k-2} \frac{L(k-1-j,f)}{j!(2\pi i)^{k-1-j}} z^j, \tag{28}$$

where, as usual, $L(s,f) = \sum_{n=1}^{\infty} a_n n^{-s}$.

Example 5.4 For instance, if

$$f(z) = \frac{1}{2}(2\pi i)^{k-1}\zeta(1-k)E_k(z) = (2\pi i)^{k-1}\left[\frac{1}{2}\zeta(1-k) + \sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right],$$

where E_k is the Eisenstein series defined in (21), then

$$f^*(z) = \frac{1}{2}(2\pi i)^{k-1}\zeta(1-k)\frac{z^{k-1}}{(k-1)!} + F_{k-1}(z)$$

equals the function $G_{2m+1}(z)$, where $k = 2m+2$, used earlier in (26). Observe that the L-series of the Eisenstein series $f(z)$ is

$$L(s,f) = (2\pi i)^{k-1}\sum_{n=1}^{\infty}\frac{\sigma_{k-1}(n)}{n^s} = (2\pi i)^{k-1}\zeta(s)\zeta(1-k+s).$$

The result (28) of Razar and Weil therefore implies that

$$z^{k-2} f^*\left(-\frac{1}{z}\right) - f^*(z) = -\sum_{j=0}^{k-2} \frac{(2\pi i)^j \zeta(k-1-j)\zeta(-j)}{j!} z^j, \tag{29}$$

and the right-hand side of (29) indeed equals the right-hand side of (27), which we obtained as a reformulation of Ramanujan's identity. In particular, we see that (28) is a vast generalization of Ramanujan's identity from Eisenstein series to other modular forms.

6 Ramanujan's Formula for $\zeta(2n+1)$ and Euler's Formula for $\zeta(2n)$ are Consequences of the Same General Theorem

Ramanujan's formula (2) for $\zeta(2n+1)$ is, in fact, a special instance of a general transformation formula for generalized analytic Eisenstein series or, in another formulation, for a vast generalization of the logarithm of the Dedekind eta function. We relate one such generalization due to the first author [5] and developed in [8], where a multitude of special cases are derived. Throughout, let z be in the upper half-plane $\mathscr{H}$, and let $V = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ be an element of $\mathrm{SL}_2(\mathbb{Z})$. That is, $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. Furthermore, let $Vz = (az+b)/(cz+d)$, as in (23). In the sequel, we assume that $c > 0$. Let r_1 and r_2 be real numbers, and let m be an integer. Define

$$A(z, -m, r_1, r_2) := \sum_{n > -r_1} \sum_{k=1}^{\infty} k^{-m-1} e^{2\pi i k(nz + r_1 z + r_2)} \tag{30}$$

and

$$H(z, -m, r_1, r_2) := A(z, -m, r_1, r_2) + (-1)^m A(z, -m, -r_1, -r_2). \tag{31}$$

We define the Hurwitz zeta function $\zeta(s, a)$, for any real number a, by

$$\zeta(s, a) := \sum_{n > -a} (n + a)^{-s}, \quad \operatorname{Re} s > 1.$$

For real numbers a, the characteristic function is denoted by $\lambda(a)$. Define $R_1 = ar_1 + cr_2$ and $R_2 = br_1 + dr_2$, where a, b, c, d are as above. Define

$$g(z, -m, r_1, r_2) := \lim_{s \to -m} \frac{\Gamma(s)}{(2\pi i)^s} \Big\{ -\lambda(r_1)(cz+d)^{-s}(e^{\pi i s}\zeta(s, r_2) + \zeta(s, -r_2)) + \lambda(R_1)(\zeta(s, R_2) + e^{-\pi i s}\zeta(s, -R_2)) \Big\}. \tag{32}$$

As customary, $B_n(x)$, $n \geq 0$, denotes the n-th Bernoulli polynomial, and $\{x\}$ denotes the fractional part of x. Let $\rho := \{R_2\}c - \{R_1\}d$. Lastly, define

$$h(z, -m, r_1, r_2) := \sum_{j=1}^{c} \sum_{k=0}^{m+2} \frac{(-1)^{k-1}(cz+d)^{k-1}}{k!(m+2-k)!} B_k\left(\frac{j - \{R_1\}}{c}\right) B_{m+2-k}\left(\frac{jd + \rho}{c}\right),$$

where it is understood that if $m + 2 < 0$, then $h(z, -m, r_1, r_2) \equiv 0$.

We are now ready to state our general transformation formula [5, p. 498], [8, p. 150].

Theorem 6.1 *If $z \in \mathscr{H}$ and m is any integer, then*

$$(cz+d)^m H(Vz,-m,r_1,r_2) = H(z,-m,R_1,R_2) + g(z,-m,r_1,r_2) + (2\pi i)^{m+1} h(z,-m,r_1,r_2). \tag{33}$$

We now specialize Theorem 6.1 by supposing that $r_1 = r_2 = 0$ and that $Vz = -1/z$, so that $c = 1$ and $d = 0$. Note that, from (30) and (31),

$$H(z,-m,0,0) = (1+(-1)^m) \sum_{k=1}^{\infty} \frac{k^{-m-1}}{e^{-2\pi i k z} - 1}. \tag{34}$$

Theorem 6.2 *If $z \in \mathscr{H}$ and m is any integer, then*

$$z^m (1+(-1)^m) \sum_{k=1}^{\infty} \frac{k^{-m-1}}{e^{2\pi i k/z} - 1} = (1+(-1)^m) \sum_{k=1}^{\infty} \frac{k^{-m-1}}{e^{-2\pi i k z} - 1} + g(z,-m) + (2\pi i)^{m+1} \sum_{k=0}^{m+2} \frac{B_k(1)}{k!} \frac{B_{m+2-k}}{(m+2-k)!} (-z)^{k-1}, \tag{35}$$

where

$$g(z,-m) = \begin{cases} \pi i - \log z, & \text{if } m = 0, \\ \{1-(-z)^m\}\zeta(m+1), & \text{if } m \neq 0. \end{cases} \tag{36}$$

Corollary 6.3 (Euler's Formula for $\zeta(2n)$) *For each positive integer n,*

$$\zeta(2n) = \frac{(-1)^{n-1} B_{2n}}{2(2n)!} (2\pi)^{2n}. \tag{37}$$

Proof Put $m = 2n-1$ in (35). Trivially, by (34), $H(z,-2m+1,0,0) = 0$. Using (36), we see that (35) reduces to

$$(1+z^{2n-1})\zeta(2n) = \frac{(2\pi)^{2n}(-1)^{n-1}}{(2n)!} \{B_1(1)B_{2n} - B_{2n}B_1 z^{2n-1}\}, \tag{38}$$

where we have used the values $B_k(1) = B_k$, $k \geq 2$, and $B_{2k+1} = 0$, $k \geq 1$. Since $B_1(1) = \frac{1}{2}$ and $B_1 = -\frac{1}{2}$, Euler's formula (37) follows immediately from (38). □

Corollary 6.4 (Ramanujan's Formula for $\zeta(2n+1)$) *Let α and β denote positive numbers such that $\alpha\beta = \pi^2$. Then, for each positive integer n,*

$$\alpha^{-n}\left\{\frac{1}{2}\zeta(2n+1)+\sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\alpha k}-1}\right\}=(-\beta)^{-n}\left\{\frac{1}{2}\zeta(2n+1)+\sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\beta k}-1}\right\} \tag{39}$$
$$-2^{2n}\sum_{k=0}^{n+1}(-1)^k\frac{B_{2k}}{(2k)!}\frac{B_{2n+2-2k}}{(2n+2-2k)!}\alpha^{n+1-k}\beta^k.$$

Proof Set $m=2n$ in (35), and let $z=i\pi/\alpha$. Recall that $\pi^2/\alpha=\beta$. If we multiply both sides by $\frac{1}{2}(-\beta)^{-n}$, we obtain (39). □

We see from Corollaries 6.3 and 6.4 that Euler's formula for $\zeta(2n)$ and Ramanujan's formula for $\zeta(2n+1)$ are natural companions, because both are special instances of the same transformation formula.

We also emphasize that in the foregoing proof, we assumed that n was a *positive* integer. Suppose that we had taken $m=-2n$, where n is a positive integer. Furthermore, if $n>1$, then the sum of terms involving Bernoulli numbers is empty. Recall that

$$\zeta(1-2n)=-\frac{B_{2n}}{2n},\quad n\geq 1. \tag{40}$$

We then obtain the following corollary.

Corollary 6.5 *Let α and β be positive numbers such that $\alpha\beta=\pi^2$. Then, for any integer $n>1$,*

$$\alpha^n\sum_{k=1}^{\infty}\frac{k^{2n-1}}{e^{2\alpha k}-1}-(-\beta)^n\sum_{k=1}^{\infty}\frac{k^{2n-1}}{e^{2\beta k}-1}=\{\alpha^n-(-\beta)^n\}\frac{B_{2n}}{4n}. \tag{41}$$

Corollary 6.5 is identical to Entry 13 in Chap. 14 of Ramanujan's second notebook [58], [10, p. 261]. It can also be found in his paper [56, p. 269], [57, p. 190] stated without proof. Corollary 6.5 is also formula (25) in Ramanujan's unpublished manuscript [59, p. 319], [1, p. 276].

Of course, by (40), we could regard (41) as a formula for $\zeta(1-2n)$, and so we would have a third companion to the formulas of Euler and Ramanujan. The first proof of (41) known to the authors is by Rao and Ayyar [60] in 1923, with Malurkar [49] giving another proof shortly thereafter in 1925. If we replace n by $2n+1$ in (41), where n is a positive integer, and set $\alpha=\beta=\pi$, we obtain the special case

$$\sum_{k=1}^{\infty}\frac{k^{4n+1}}{e^{2\pi k}-1}=\frac{B_{4n+2}}{4(2n+1)}, \tag{42}$$

which is due much earlier to Glaisher [24] in 1889. The formula (42) can also be found in Sect. 13 of Chap. 14 in Ramanujan's second notebook [58], [10, p. 262].

There is yet a fourth companion. If we set $m = -2$ in (35), proceed as in the proof of Corollary 6.4, and use (40), we deduce the following corollary, which we have previously recorded as Entry 2.3, and which can be thought of as "a formula for $\zeta(-1)$."

Corollary 6.6 *Let α and β be positive numbers such that $\alpha\beta = \pi^2$. Then*

$$\alpha \sum_{k=1}^{\infty} \frac{k}{e^{2\alpha k} - 1} + \beta \sum_{k=1}^{\infty} \frac{k}{e^{2\beta k} - 1} = \frac{\alpha + \beta}{24} - \frac{1}{4}. \tag{43}$$

If $\alpha = \beta = \pi$, (43) reduces to

$$\sum_{k=1}^{\infty} \frac{k}{e^{2\pi k} - 1} = \frac{1}{24} - \frac{1}{8\pi}. \tag{44}$$

Both the special case (44) and the more general identity (43) can be found in Ramanujan's notebooks [58, vol. 1, p. 257, no. 9; vol. 2, p. 170, Corollary 1], [10, pp. 255–256]. However, in 1877, Schlömilch [66], [67, p. 157] apparently gave the first proof of both (44) and (43).

In conclusion of Sect. 6, we point out that several authors have proved general transformation formulas from which Ramanujan's formula (39) for $\zeta(2n + 1)$ can be deduced as a special case. However, in most cases, Ramanujan's formula was not explicitly recorded by the authors. General transformation formulas have been proved by Guinand [28, 29], Chandrasekharan and Narasimhan [18], Apostol [3, 4], Mikolás [50], Iseki [31], Bodendiek [14], Bodendiek and Halbritter [15], Glaeske [22, 23], Bradley [17], Smart [70], and Panzone, Piovan, and Ferrari [53]. Guinand [28, 29] did state (39). Due to a miscalculation, $\zeta(2n+1)$ did not appear in Apostol's formula [3], but he later [4] realized his mistake and so discovered (39). Also, recall that in Sect. 3, we mentioned the very general transformation formula for multiple Barnes zeta functions by Komori, Matsumoto, and Tsumura [40] that contains Ramanujan's formula (39) for $\zeta(2n + 1)$ as a special case. Lastly, in the beautiful work of Katsurada [38, 39] on asymptotic expansions of q-series, Ramanujan's formula (2) arises as a special case.

We also have not considered further formulas for $\zeta(2n+1)$ that would arise from the differentiation of, e.g., (33) and (35) with respect to z; see [8].

7 The Associated Polynomials and Their Roots

In this section we discuss the polynomials that are featured in Ramanujan's formula (2) and discuss several natural generalizations. These polynomials have received considerable attention in the recent literature. In particular, several papers focus on the location of zeros of these polynomials. We discuss some of the

developments starting with Ramanujan's original formula and indicate avenues for future work.

Following [30] and [51], define the *Ramanujan polynomials*

$$R_{2m+1}(z) = \sum_{n=0}^{m+1} \frac{B_{2n}}{(2n)!} \frac{B_{2m-2n+2}}{(2m-2n+2)!} z^{2n} \tag{45}$$

to be the polynomials appearing on the right-hand side of Ramanujan's formula (2) and (19). The discussion in Sect. 5 demonstrates that the Ramanujan polynomials are, essentially, the odd parts of the period polynomials attached to Eisenstein series of level 1. In [51], Murty, Smyth, and Wang proved the following result on the zeros of the Ramanujan polynomials.

Theorem 7.1 *For $m \geq 0$, all nonreal zeros of the Ramanujan polynomials $R_{2m+1}(z)$ lie on the unit circle.*

Moreover, it is shown in [51] that, for $m \geq 1$, the polynomial $R_{2m+1}(z)$ has exactly four real roots and that these approach $\pm 2^{\pm 1}$, as $m \to \infty$.

Example 7.2 As described in [51], one interesting consequence of this result is that there exists an algebraic number $\alpha \in \mathscr{H}$ with $|\alpha| = 1$, $\alpha^{2m} \neq 1$, such that $R_{2m+1}(\alpha) = 0$ and, consequently,

$$\frac{\zeta(2m+1)}{2} = \frac{F_{2m+1}(\alpha) - \alpha^{2m} F_{2m+1}\left(-\frac{1}{\alpha}\right)}{\alpha^{2m} - 1}, \tag{46}$$

where F is as in (18). In other words, the odd zeta values can be expressed as the difference of two Eichler integrals evaluated at special algebraic values. An extension of this observation to Dirichlet L-series is discussed in [13]. Equation (46) follows directly from Ramanujan's identity, with $z = \alpha$, in the form (19). Remarkably, Eq. (46) gets close to making a statement on the transcendental nature of odd zeta values: Gun, Murty, and Rath prove in [30] that, as α ranges over *all* algebraic values in $\mathscr{H}$ with $\alpha^{2m} \neq 1$, the set of values obtained from the right-hand side of (46) contains at most one algebraic number.

As indicated in Sect. 5, the Ramanujan polynomials are the odd parts of the period polynomials attached to Eisenstein series (or, rather, period functions; see Remark 5.2 and [74]). On the other hand, it was conjectured in [42], and proved in [43], that the full period polynomial is in fact *unimodular*, that is, all of its zeros lie on the unit circle.

Theorem 7.3 *For $m > 0$, all zeros of the polynomials*

$$R_{2m+1}(z) + \frac{\zeta(2m+1)}{(2\pi i)^{2m+1}}(z^{2m+1} - z)$$

lie on the unit circle.

An analogue of Theorem 7.1 for cusp forms has been proved by Conrey, Farmer, and Imamoglu [19], who show that, for any Hecke cusp form of level 1, the odd part of its period polynomial has trivial zeros at 0, $\pm 2^{\pm 1}$ and all remaining zeros lie on the unit circle. Similarly, El-Guindy and Raji [20] extended Theorem 7.3 to cusp forms by showing that the full period function of any Hecke eigenform of level 1 has all its zeros on the unit circle.

Remark 7.4 In light of (27) it is also natural to ask whether the polynomials

$$p_m(z) := \frac{\zeta(2m+1)}{2}(1-z^{2m}) - \frac{(2\pi i)^{2m+1}}{2}\sum_{n=1}^{m}\frac{B_{2n}}{(2n)!}\frac{B_{2m-2n+2}}{(2m-2n+2)!}z^{2n-1}$$

are unimodular. Numerical evidence suggests that these period polynomials $p_m(z)$ are indeed unimodular. Moreover, let $p_m^-(z)$ be the odd part of $p_m(z)$. Then, the polynomials $p_m^-(z)/z$ also appear to be unimodular. It seems reasonable to expect that these claims can be proved using the techniques used for the corresponding results in [51] and [43]. We leave this question for the interested reader to pursue.

Very recently, Jin, Ma, Ono, and Soundararajan [32] established the following extension of the result of El-Guindy and Raji to cusp forms of any level.

Theorem 7.5 *For any newform $f \in S_k(\Gamma_0(N))$ of even weight k and level N, all zeros of the period polynomial, given by the right-hand side of (28), lie on the circle $|z| = 1/\sqrt{N}$.*

Extensions of Ramanujan's identity, and some of its ramifications, to higher level are considered in [13]. Numerical evidence suggests that certain polynomials arising as period polynomials of Eisenstein series again have all their roots on the unit circle. Especially in light of the recent advance made in [32], it would be interesting to study period polynomials of Eisenstein series of any level more systematically. Here, we only cite one conjecture from [13], which concerns certain special period polynomials, conveniently rescaled, and suggests that these are all unimodular.

Conjecture 7.6 *For nonprincipal real Dirichlet characters χ and ψ, the generalized Ramanujan polynomial*

$$R_k(z;\chi,\psi) = \sum_{s=0}^{k}\frac{B_{s,\chi}}{s!}\frac{B_{k-s,\psi}}{(k-s)!}\left(\frac{z-1}{M}\right)^{k-s-1}(1-z^{s-1}) \tag{47}$$

is unimodular, that is, all its roots lie on the unit circle.

Here, $B_{n,\chi}$ are the generalized Bernoulli numbers, which are defined by

$$\sum_{n=0}^{\infty}B_{n,\chi}\frac{x^n}{n!} = \sum_{a=1}^{L}\frac{\chi(a)xe^{ax}}{e^{Lx}-1}, \tag{48}$$

if χ is a Dirichlet character modulo L. If χ and ψ are both nonprincipal, then $R_k(z;\chi,\psi)$ is indeed a polynomial. On the other hand, as shown in [13], if

$k > 1$, then $R_{2k}(z;1,1) = R_{2k-1}(z)/z$, that is, despite their different appearance, the generalized Ramanujan polynomials reduce to the Ramanujan polynomials, introduced in (45), when $\chi = 1$ and $\psi = 1$. For more details about this conjecture, we refer to [13].

8 Further History of Proofs of (2)

In this concluding section, we shall offer further sources where additional proofs of Ramanujan's formula for $\zeta(2n+1)$ can be found. We emphasize that two papers of Grosswald [25, 27], each proving formulas from Ramanujan's second notebook, with the first giving a proof of (2), stimulated the first author and many others to seriously examine the content of Ramanujan's notebooks [58]. Katayama [36, 37], Riesel [63], Rao [61], Vepštas [71], Zhang [75], and Zhang and Zhang [76] have also developed proofs of (2). Ghusayni [21] has used Ramanujan's formula for $\zeta(2n+1)$ for numerical calculations.

For further lengthy discussions, see the first author's book [10, p. 276], his survey paper [6], his account of Ramanujan's unpublished manuscript [12], and his fourth book with Andrews [1, Chap. 12] on Ramanujan's lost notebook. These sources also contain a plethora of references to proofs of the third and fourth companions to Ramanujan's formula for $\zeta(2n+1)$. Another survey has been given by Kanemitsu and Kuzumaki [33].

Also, there exist a huge number of generalizations and analogues of Ramanujan's formula for $\zeta(2n+1)$. Some of these are discussed in Berndt's book [10, p. 276] and his paper [9]. Lim [46–48] has established an enormous number of beautiful identities in the spirit of the identities discussed in the present survey. Kanemitsu, Tanigawa, and Yoshimoto [34, 35], interpreting Ramanujan's formula (2) as a modular transformation, derived further formulas for $\zeta(2n+1)$, which they have shown lead to a rapid calculation of $\zeta(3)$ and $\zeta(5)$, for example.

In Sects. 4 and 5 we have seen that Ramanujan's formula can be viewed as describing the transformation laws of the modular Eisenstein series E_{2k}, where $k > 1$, of level 1 (that is, with respect to the full modular group), the quasimodular Eisenstein series E_2 as well as their Eichler integrals. We refer to [13] for extensions of these results to higher level.

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Towards a Fractal Cohomology: Spectra of Polya–Hilbert Operators, Regularized Determinants and Riemann Zeros

Tim Cobler and Michel L. Lapidus

Abstract Emil Artin defined a zeta function for algebraic curves over finite fields and made a conjecture about them analogous to the famous Riemann hypothesis. This and other conjectures about these zeta functions would come to be called the Weil conjectures, which were proved by Weil in the case of curves and eventually, by Deligne in the case of varieties over finite fields. Much work was done in the search for a proof of these conjectures, including the development in algebraic geometry of a Weil cohomology theory for these varieties, which uses the Frobenius operator on a finite field. The zeta function is then expressed as a determinant, allowing the properties of the function to relate to the properties of the operator. The search for a suitable cohomology theory and associated operator to prove the Riemann hypothesis has continued to this day. In this paper we study the properties of the derivative operator $D = \frac{d}{dz}$ on a particular family of weighted Bergman spaces of entire functions on $\mathbb{C}$. The operator D can be naturally viewed as the "infinitesimal shift of the complex plane" since it generates the group of translations of $\mathbb{C}$. Furthermore, this operator is meant to be the replacement for the Frobenius operator in the general case and is used to construct an operator associated with any given meromorphic function. With this construction, we show that for a wide class of meromorphic functions, the function can be recovered by using a regularized determinant involving the operator constructed from the meromorphic function. This is illustrated in some important special cases: rational functions, zeta functions of algebraic curves (or, more generally, varieties) over finite fields, the Riemann zeta function, and culminating in a quantized version of the Hadamard factorization theorem that applies to any entire function of finite order. This shows that all of the information about the given meromorphic function is encoded into the special operator we constructed. Our construction is motivated in part by work of Herichi and the second author on the infinitesimal shift of the real line (instead of the complex plane) and the associated spectral operator, as well as by earlier work and

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H. Montgomery et al. (eds.), *Exploring the Riemann Zeta Function*,
DOI 10.1007/978-3-319-59969-4_3

conjectures of Deninger on the role of cohomology in analytic number theory, and a conjectural "fractal cohomology theory" envisioned in work of the second author and of Lapidus and van Frankenhuijsen on complex fractal dimensions.

1 Introduction

Riemann's famous paper [31] opened up the use of complex analysis to study the prime numbers. This approach has yielded many great results in number theory, including, but certainly not limited to, the Prime Number Theorem. Riemann also made his well-known conjecture that stands to this very day. We will refer to this as (RH) in this paper.

Conjecture (RH) The only nontrivial zeros of $\zeta(s)$ occur when s satisfies $\Re(s) = \frac{1}{2}$.

For further reading about $\zeta(s)$, see [30, 37], and [13]. However, despite (RH) remaining unsolved over 150 years after it was made, there is an analogue for zeta functions of algebraic varieties over finite fields that has been proven. The development of this theory also introduced new techniques to number theory. We will begin with a short history of this result.

1.1 The Weil Conjectures

Using the Euler product representation of the Riemann zeta function in terms of the rational primes,

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1}, \tag{1}$$

as a template, it is possible to define the zeta function of an algebraic curve over a finite field as follows.

Definition 1 Let Y be a smooth, geometrically connected curve over $\mathbb{F}_q$, the finite field with q elements. Then the zeta function of Y is given by

$$\zeta_Y(s) = \prod_{y \in |Y|} (1 - |k_y|^{-s})^{-1}, \tag{2}$$

where $|Y|$ is the set of closed points of Y and $|k_y|$ is the size of the residue field of y.

This formulation of the zeta function of an algebraic curve over a finite field shows the analogy with Riemann's zeta function, but we will prefer the following equivalent expression $\zeta_Y(s) = \exp\left(\sum_{n=1}^{\infty} \frac{Y_n}{n} q^{-ns}\right)$, where Y_n is the number of points of Y defined over $\mathbb{F}_{q^n}$, the degree n extension of $\mathbb{F}_q$. The study of these zeta functions began in 1924 in Emil Artin's PhD thesis [1]. These were further studied

by Schmidt, who proved, in 1931, that $\zeta_Y(s)$ was a rational function of q^{-s} in [34], and Hasse, who showed, in 1934, in [18], that if Y is an elliptic curve, then the zeros of $\zeta_Y(s)$ satisfy $\Re(s) = \frac{1}{2}$. Thus, the corresponding version of (RH) holds for these zeta functions of elliptic curves over finite fields. Furthering this idea, Weil then proved, in 1946–1948, that this same version of (RH) holds for algebraic curves of arbitrary genus and for abelian varieties in [40]. (See also [38, 39] and [41].) Below we present a sketch of some of the ideas contained in a modern proof of these results, which are based on Weil's ideas, and will motivate the work contained in this paper.

First, a sequence of the so-called Weil cohomology groups for the curve Y are formed, in particular H^0, H^1, H^2 are the only nontrivial groups, with $\dim H^0 = \dim H^2 = 1$ and $\dim H^1 = 2g$ where g denotes the genus of Y. Then the Frobenius map F which sends $y \to y^q$ acts on the space $\mathbb{F}_{q^n}$ for any n and therefore induces a morphism of the curve Y over $\overline{\mathbb{F}_q}$ (the algebraic closure of $\mathbb{F}_q$) as well as, in fact, also induces a linear map on the cohomology groups $F^* : H^j \to H^j$, for $j \in \{0, 1, 2\}$.

Next, consider the Lefschetz fixed point formula from topology.

Theorem 1 (Lefschetz Fixed Point Formula) *Let Y be a closed smooth manifold and let $f : Y \to Y$ be a smooth map with all fixed points nondegenerate. Then $\sum_{j=0}^{\infty}(-1)^j Tr(f^*|H^j)$ is equal to the number of fixed points of f.*

Note that in Theorem 1, since Y is finite-dimensional, only finitely many of the cohomology spaces H^j are nontrivial.

We apply the topological version of this result to the nth power of the Frobenius map, F^n, whose fixed points are exactly the points on the curve Y defined over $\mathbb{F}_{q^n}$. That is, all those points with every coordinate in $\mathbb{F}_{q^n}$. This gives

$$\sum_{j=0}^{2}(-1)^j \mathrm{Tr}(F^{*^n}|H^j) = Y_n, \tag{3}$$

where $F^*|H^j$ (for $j = 0, 1, 2$) denotes the linear operator induced on the cohomology space H^j by the Frobenius morphism F.

To proceed further, we need the next result from linear algebra.

Theorem 2 *If f is an endomorphism of a finite-dimensional vector space V, then for $|t|$ sufficiently small,* $\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}t^n \mathrm{Tr}(f^n|V)\right) = \det(I - f \cdot t|V)^{-1}$.

Applying this result to the Frobenius operator F, we can proceed with the following calculation:

$$\begin{aligned}\zeta_Y(s) &= \exp\left(\sum_{n=1}^{\infty}\frac{Y_n}{n}q^{-ns}\right)\\ &= \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\sum_{j=0}^{2}(-1)^j \mathrm{Tr}(F^{*^n}|H^j)q^{-ns}\right)\end{aligned}$$

$$= \prod_{j=0}^{2} \left(\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}(F^{*^n} | H^j) q^{-ns} \right) \right)^{(-1)^j}$$

$$= \prod_{j=0}^{2} \left(\det(I - F^* q^{-s} | H^j) \right)^{(-1)^{j+1}}$$

$$= \frac{\det(I - F^* q^{-s} | H^1)}{\det(I - F^* q^{-s} | H^0) \det(I - F^* q^{-s} | H^2)}. \tag{4}$$

This enables us to express the zeta function of a curve Y as an alternating product of characteristic polynomials of the Frobenius operators, or more precisely, of determinants of $I - q^{-s}F^*$ over the cohomology spaces. Since these spaces are finite-dimensional, this equation further shows that $\zeta(Y, s)$ is a rational function of q^{-s}, which yields Schmidt's result. We also see that the zeros of $\zeta_Y(s)$ are given from the eigenvalues of the operator F^* on H^1, while the poles are given from the eigenvalues on H^0 and H^2. To complete the proof, it was shown by Weil using the intersection theory of divisors to show that the intersection is positive definite that the eigenvalues of F^* on H^j have absolute value $q^{\frac{j}{2}}$ and thus the zeros of $\zeta(Y, s)$ satisfy $\Re(s) = \frac{1}{2}$.

Weil then conjectured that all of the above and more could be generalized to any non-singular, projective variety of dimension d, defined over $\mathbb{F}_q$. About a decade later, Alexander Grothendieck announced he would be revamping algebraic geometry with the goal of proving these Weil conjectures. Several attempts to construct a proper "Weil cohomology" were incomplete, but eventually these provided the key idea to the proof of the Weil conjectures. Grothendieck even came up with more general conjectures based on this study of what properties a "Weil cohomology" must possess. The version of (RH) sought after would then follow from these. Some of his work outlining these ideas are [14, 15], and [16]. However, Pierre Deligne, a student of Grothendieck, would go on to prove, in 1973, this version of (RH) without proving Grothendieck's "standard conjectures," which are still unproven today. See [8] and [9] for Deligne's work. Thus, Weil's conjectures were completed as a result of the introduction of, or at the very least, expansion of, the use of topology and cohomology in number theory. For a more complete history of the Weil conjectures, see [12, 24] and [29].

1.2 *Polya–Hilbert Operators and a Cohomology Theory in Characteristic Zero*

As seen in the previous section, the Frobenius operator became fundamental to the resolution of the version of (RH) dealing with algebraic varieties (or even with curves) over finite fields. The eigenvalues of this operator on different cohomology

groups gave us the zeros and poles of the zeta function of the variety. If such an operator could be found for the Riemann zeta function, then perhaps this work would extend and help one to prove (RH). However, if you instead consider the function $\zeta(\frac{1}{2} + it)$ as a function of t, then (RH) is equivalent to all the nontrivial (or critical) zeros of this function being real. This then leads into what is known as the Polya–Hilbert conjecture.

Conjecture (Polya–Hilbert Conjecture) The critical zeros of $\zeta(\frac{1}{2} + it)$ correspond to the eigenvalues of an unbounded self-adjoint operator on a suitable Hilbert space.

Since then, motivated in part by the above reformulation, many physicists, mathematicians, and mathematical physicists have been looking for a convincing physical reason why (RH) should be true. In particular, it has been conjectured by Michael Berry in [3] (and several other papers) that a trace formula for a suitable (classically chaotic) quantum-mechanical Hamiltonian could formalize this connection between the spectrum of an operator and the Riemann zeros. See also [4] for a discussion of these ideas. In fact, Alain Connes, in [6], conjectured the existence of a suitable noncommutative version of such a trace formula. However, as in every other approach to proving (RH), the search for the correct way to make this potential approach work continues to this day.

Building on Alexander Grothendieck's ideas, Christopher Deninger has postulated in [10, 11] (and other papers) that the cohomology theory used to prove the Weil conjectures could be extended to eventually prove the Riemann hypothesis. In particular, he envisions a cohomology theory of algebraic schemes over $Spec(\mathbb{Z})$ that would conjecturally help prove the Riemann hypothesis and solve other important problems in analytic number theory. In his papers, he lays out some of the difficulties in doing so as well as some of the properties that such a theory would need to satisfy.

We also mention that Shai Haran [17] has obtained interesting trace formulas yielding Weil's explicit formula: that is, of Weil's interpretation in [42] of Riemann's explicit formula [13, 25, 28, 30, 31, 37].

1.3 Fractal Cohomology

All of the previous ideas as well as separate connections between fractals and the Riemann zeta function $\zeta(s)$ discussed in [27] and [28] motivated the second author to pursue a fractal cohomology to try to tie together all of the ideas presented so far. The text [25] outlines his ideas for how the theory of fractals might give information about the Riemann zeta function. See also Sect. 12.4 of [28] for a discussion of the main properties that such a fractal cohomology theory should satisfy, by analogy with the case of varieties over finite fields and self-similar strings.

In search of the elusive "Frobenius operator in characteristic 0," the second author worked with H. Herichi to develop a "Quantized Number Theory" in [20–23, 26]. There, they used an operator they denoted ∂, which was the derivative operator on a suitable family of Hilbert spaces. This operator had many nice properties,

including being a generator for the infinitesimal shift group on these spaces as well as having a spectrum consisting of a single vertical line in the complex plane. This allowed them to focus on the values of $\zeta(s)$ on $\Re(s) = c$ for $c \in (0, \frac{1}{2})$ or for $c \in (\frac{1}{2}, 1)$ and obtain a reformulation of (RH) within this theory. This involved studying an operator-valued version of $\zeta(s)$, which they called a quantized zeta function. An overview of these ideas and results can be found in [26], while a detailed exposition of the theory is provided in [23].

This paper then continues this search of an appropriate substitute for Frobenius in characteristic zero. In an attempt to further localize the spectrum of the derivative operator, we turn to a family of weighted Bergman spaces, which provide the basis for our construction. We will begin by recalling some needed functional analysis building up to the regularized determinants that we will need. Then we discuss the family of Bergman spaces and the needed properties of the derivative operator on them, which allows our construction to work. At this point, we will detail our construction to create a Frobenius replacement. This provides a general framework to find a substitute for the Frobenius operator which will be shown to apply to any entire function of finite order as well as certain meromorphic functions of interest such as $\zeta(s)$. However, there is still much to be done. We do not have a true cohomology theory as we do not have a suitable notion for how to define the geometry in our context. We will finish with a discussion of what is lacking from this theory and where to go from here.

2 Background

This section loosely follows [36] in developing the necessary theory for trace ideals and regularized determinants to be used in this paper. See [35, 36] for detailed historical notes and references, along with a discussion of the many contributions to this subject.

To describe what trace ideals are, we recall some standard facts about compact operators on a separable Hilbert space H.

Theorem 3 *Let A be a compact operator on H. Then there are orthonormal sets $\{\psi_n\}$ and $\{\phi_n\}$ and positive real numbers $\mu_n(A)$, with $\mu_1(A) \geq \mu_2(A) \geq \cdots$, such that $A = \sum_n \mu_n(A)(\psi_n, \cdot)\phi_n$. Moreover, the numbers $\mu_n(A)$ are uniquely determined.*

The positive real numbers $\mu_n(A)$ from the previous theorem are called the *singular values* of A. We can actually describe $\{\mu_n(A)\}$ in another way. Given an operator A, the operator A^*A is a nonnegative operator, so that $|A| := \sqrt{A^*A}$ makes sense. The $\mu_n(A)'s$ are exactly the (nonzero) eigenvalues of $|A|$. We can now turn to Calkin's theory of operator ideals. We begin by setting up a relationship between ideals in $B(H)$ and certain sequence spaces.

Definition 2 Fix an orthonormal set $\{\phi_n\}$ in H. Given an ideal $J \neq B(H)$; we define the *sequence space associated with J* by

$$S(J) = \left\{ a = (a_1, a_2, \ldots) | \sum_n a_n(\phi_n, \cdot)\phi_n \in J \right\} . \tag{5}$$

On the other hand, given a sequence space $\mathbf{s}$, let $I(\mathbf{s})$ be the family of compact operators A with $(\mu_1(A), \mu_2(A), \ldots) \in \mathbf{s}$.

In order for this correspondence between sequence spaces and ideals to be one-to-one, we need to restrict our sequence spaces to Calkin spaces. We then need the following operator on sequences.

Definition 3 Given an infinite sequence, (a_n), of numbers with $a_n \to 0$ as $n \to \infty$, a_n^* is the sequence defined by $a_1^* = \max_i |a_i|$, $a_1^* + a_2^* = \max_{i \neq j}(|a_i| + |a_j|)$, etc. Thus $a_1^* \geq a_2^* \geq \cdots$, and the sets of a_i^* and $|a_i|$ are identical, counting multiplicities. This operator allows us to make the following definition.

Definition 4 A *Calkin space* is a vector space, $\mathbf{s}$, of sequences (a_n) with $\lim_{n\to\infty} a_n = 0$, and the so-called Calkin property: $a \in \mathbf{s}$ and $b_n^* \leq a_n^*$ implies $b \in \mathbf{s}$.

With these definitions in mind, we can use the following theorem to see a relation between two-sided ideals and Calkin spaces.

Theorem 4 ([36]) *If $\boldsymbol{s}$ is a Calkin space, then $I(\boldsymbol{s})$ is a two-sided ideal of operators and $S(I(\boldsymbol{s})) = \boldsymbol{s}$. Furthermore, if J is a two-sided ideal, then $S(J)$ is a Calkin space and $I(S(J)) = J$.*

We will now use this relation to define the ideals in the space of compact operators that we will be working with.

Definition 5 A compact operator A is said to be in the *trace ideal* J_p, for some $p \geq 1$, if $\sum_n \mu_n(A)^p < \infty$. That is, J_p is the ideal which is associated with the Calkin space l^p. An element A of J_1 is called a *trace class operator*. For $A \in J_1$, we define $\mathrm{Tr}(A) = \sum_n (\phi_n, A\phi_n)$ for any choice of orthonormal basis $\{\phi_n\}$. If $A \in J_2$, then we say that A is *Hilbert–Schmidt*.

Trace class operators, A, are precisely those operators for which the expression $\mathrm{Tr}(A) = \sum_n (\phi_n, A\phi_n)$ is absolutely convergent and independent of the choice of orthonormal basis. Similarly, Hilbert–Schmidt operators are those for which $\sum_n (A\phi_n, A\phi_n) = \|A\phi_n\|^2$ is convergent and independent of the choice of orthonormal basis. If A is a trace class operator, then there is a method to define a so-called *Fredholm determinant*, $\det(I + zA)$, which defines an entire function on $\mathbb{C}$. Operators of the form $I + zA$ for a trace class operator A are called *Fredholm*. This determinant can be defined in several equivalent ways. We list them here for trace class A and $z \in \mathbb{C}$:

$$\det(I + zA) := e^{\mathrm{Tr}(\log(I+zA))} \tag{6}$$

for small $|z|$ and then analytically continued to the whole complex plane,

$$\det(I + zA) = \sum_{k=0}^{\infty} z^k \mathrm{Tr}(\wedge^k(A)) \tag{7}$$

with $\wedge^k(A)$ defined in terms of alternating algebras, and

$$\det(I + zA) = \prod_{k=1}^{N(A)} (1 + z\lambda_k(A)), \tag{8}$$

where the complex numbers $\lambda_k(A)$ are the nonzero eigenvalues of A and $N(A)$ is the number of such eigenvalues, which can be infinite. In the latter case, the corresponding infinite product is convergent.

A discussion concerning which of the above equations should be taken as a definition and which are to be proven appears briefly in Chap. 3 of [36] and in more detail in [35]. For the work here, (8) will be the most convenient choice. One thing to note at this time though is that $\det(I + zA)$ does define an entire function by any of the above definitions, when A is trace class. This then shows why one cannot hope to recover a meromorphic function by simply taking the determinant of a suitable operator without taking the quotient of such determinants as was seen in the discussion of the Weil conjectures.

Although some of the operators we will consider will not be trace class, they will at least be in one of the other trace ideals J_n, for some $n \in \mathbb{N}$. In this case, we can define a regularized determinant that will allow us to get a determinant formula for the operator. We start by considering an expression of the form $\det(I + zA)e^{-zTr(A)}$. For trace class operators A, both $\det(I + zA)$ and $e^{-zTr(A)}$ are convergent, but for Hilbert–Schmidt operators neither is necessarily well defined. And yet, when you consider the two factors together as a possibly infinite product over the eigenvalues of A,

$$\prod_{k=1}^{N(A)} ((1 + z\lambda_k(A)) \exp(-\lambda_k(A)z)\,,$$

the combined term does converge for Hilbert–Schmidt operators. This idea can in fact be extended to get a convergent infinite product expression for operators in any J_n, which will be called the *regularized determinant* of A. First we need a lemma.

Lemma 1 ([36]) *For $A \in B(H)$, let*

$$R_n(A) = \left[(I + A) \exp\left(\sum_{j=1}^{n-1} (-1)^j j^{-1} A^j \right) \right] - I. \tag{9}$$

Then if $A \in J_n$, we have $R_n(A) \in J_1$.

This associates a trace class operator to any given $A \in J_n$ and allows us to define the *regularized determinant* of A as follows:

Definition 6 ([36]) For $A \in J_n$, define $\det_n(I + A) = \det(I + R_n(A))$.

Note that this definition implies that $\det_1(I + A) = \det(I + A)$, the usual Fredholm determinant. We will use these two notations interchangeably from here on. Also with this definition, we can now give a very similar product formula for the regularized determinant of a Hilbert–Schmidt operator, with each term having an exponential factor to help convergence along with some other interesting properties. This corresponds to the $n = 2$ case of the following result.

Theorem 5 ([36]) *For $A \in J_n$, we have*

$$\det_n(I + \mu A) = \prod_{k=1}^{N(A)} \left[(1 + \mu\lambda_k(A)) \exp\left(\sum_{j=1}^{n-1} (-1)^j j^{-1} \lambda_k(A)^j \mu^j \right) \right]. \tag{10}$$

These regularized determinants are related to the usual Fredholm determinant of $1 + A$ for trace class operators A in the following fashion.

Theorem 6 ([36]) *For $A \in J_1$, we have*

$$\det_n(I + \mu A) = \det(I + A) \exp\left(\sum_{j=1}^{n-1} (-1)^j j^{-1} \mathrm{Tr}(A^j) \right). \tag{11}$$

These "regularized determinants," just as the Fredholm determinants, define an entire function and will be key to the precise formulation of our results. This will be because our construction will not always create a trace class operator for which the standard Fredholm determinant would apply. This will be the case, in particular, for the Riemann zeta function, for which the regularized determinant $\det_2$ will be needed; see Theorems 27 and 28 in Sect. 5.4 below. However, we will show that for any entire function of finite order and for many meromorphic functions, our construction will give an operator that is at least in some J_p and thus the regularized determinant will apply to it.

3 Derivative Operator on Weighted Bergman Spaces

The search for an operator to possibly take the place of the Frobenius in the proof of the Weil conjectures led us to consider the derivative operator. A treatment examining the derivative operator on $L^2(\mathbb{R}, e^{-2ct}dt)$ and its use to create a "quantized number theory" can be found in the research monograph [23], as well as in the accompanying articles [20–22] and [26].

This paper takes a different direction with the derivative operator. We begin by following the treatment in [2] in developing properties of the derivative operator on a certain family of weighted Bergman spaces. We will then continue beyond their

results and use all of this to create an operator that might properly take the place of the Frobenius. We begin by recalling the definitions of the spaces we will be working with. (See, e.g., [19], for a general reference about Bergman spaces.)

Definition 7 We define a *weight function* to be a positive continuous function w on $\mathbb{C}$. Then, for $1 \leq p \leq \infty$, we define the *weighted L^p spaces* to be $L^p_w(\mathbb{C})$, the space of functions on $\mathbb{C}$ such that $fw \in L^p(\mathbb{C}, d\lambda)$, where λ is the Lebesgue measure on $\mathbb{R}^2$, and equipped with the norm $\|f\|_{L^p_w} = \|fw\|_{L^p(\mathbb{R}^2)}$. Next, denote by B^p_w the subspace of entire functions in L^p_w; then, B^p_w is called a *weighted Bergman space of entire functions*.

Note that the convention above for functions $f \in L^p_w$ would be those for which $\int_{\mathbb{C}} |f|^p w^p d\lambda < \infty$ instead of $\int_{\mathbb{C}} |f|^p w d\lambda < \infty$. Then we have the following basic fact about these spaces.

Theorem 7 *For $p \geq 1$, B^p_w is a closed subspace of L^p_w and hence is a Banach space. Also, for $p = 2$, B^2_w is a Hilbert space.*

Now we consider the differential operator $D = \frac{d}{dz}$ on the space B^p_w and examine its properties; including for particular choices of w and p.

Consider the following types of weight functions: $w(z) = e^{-\phi(|z|)}$, where ϕ is a nonnegative, concave, monotone (i.e., nondecreasing), subadditive function on $\mathbb{R}_+ = [0, \infty)$ such that $w(0) = 0$ and

$$\lim_{t \to +\infty} \frac{\phi(t)}{\log t} = +\infty. \tag{12}$$

We next define

$$a = \lim_{t \to +\infty} \frac{\phi(t)}{t}. \tag{13}$$

We then have the following results in this situation (with $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$).

Theorem 8 ([2]) *Let $1 \leq p \leq \infty$ and w be a weight function with constant a, as in (13) above. Then,*

1) The differentiation operator $D = \frac{d}{dz}$ is a bounded linear operator on B^p_w;
2) For all $r > 0$, and for $n \in \mathbb{N}_0$, we have the bound $\|D^n\| \leq n! r^{-n} e^{\phi(r)}$.

Proof We will prove 2) noting that 1) follows from it. Suppose that $f \in B^w_p$ and $r > 0$. Cauchy's formula for the n^{th} derivative of f reads as

$$D^n f(z_0) = \frac{n!}{2\pi i} \int_{|z|=r} \frac{f(z_0 + z)}{z^{n+1}} dz. \tag{14}$$

We now consider the case $p = \infty$. Let $z_0, z \in \mathbb{C}$ with $|z| = r$. Then since ϕ is subadditive and monotonic, we have: $\phi(|z_0 + z|) \leq \phi(|z_0| + |z|) \leq \phi(|z_0|) + \phi(|z|)$. Also, $\|f\|_{\infty,w} = \sup_{z \in \mathbb{C}} |f(z)| e^{-\phi(|z|)} \geq |f(z_0 + z)| e^{-\phi(|z_0 + z|)}$ by the definition of

the norm. This leads to: $|f(z_0+z)| \le \|f\|_{\infty,w} e^{\phi(|z_0+z|)} \le \|f\|_{\infty,w} e^{\phi(|z_0|)} e^{\phi(|z|)}$. Then by (14), we have for any $z \in \mathbb{C}$ that

$$|D^n f(z)| \le n! r^{-n} \sup_{|z|=r} |f(z_0+z)| \le n! r^{-n} \|f\|_{\infty,w} e^{\phi(|z|)} e^{\phi(r)}.$$

Thus $\|D^n f\|_{\infty,w} = \sup_{z\in\mathbb{C}} |D^n f(z)| e^{-\phi(|z|)} \le n! r^{-n} e^{\phi(r)} \|f\|_{\infty,w}$. Therefore, we have that $\|D^n\| \le n! r^{-n} e^{\phi(r)}$. Next, we turn to the case $1 \le p < \infty$. Let $z \in \mathbb{C}$. Applying Hölder's inequality in (14) yields

$$|D^n f(z)| \le \frac{n!}{(2\pi)^{\frac{1}{p}} r^n} \left(\int_0^{2\pi} |f(z+re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}.$$

This leads to

$$\int_{\mathbb{C}} |D^n f(z)|^p e^{-p\phi(|z|)} d\lambda(z) \le \frac{n!^p}{2\pi r^{pn}} \int_0^{2\pi} \left(\int_{\mathbb{C}} |f(z+re^{i\theta})|^p e^{-p\phi(|z|)} d\lambda(z) \right) d\theta.$$

By making a change of variable, we can rewrite this as

$$\int_{\mathbb{C}} |D^n f(z)|^p e^{-p\phi(|z|)} d\lambda(z) \le \frac{n!^p}{2\pi r^{pn}} \int_0^{2\pi} \left(\int_{\mathbb{C}} |f(z)|^p e^{-p\phi(|z-re^{i\theta}|)} d\lambda(z) \right) d\theta. \tag{15}$$

Using the triangle inequality for ϕ,

$$|\phi(|z|) - \phi(|z-re^{i\theta}|)| \le \phi(|re^{i\theta}|) = \phi(r),$$

in the inner integral on the right-hand side of (15), we obtain that

$$\begin{aligned} \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(|z-re^{i\theta}|)} d\lambda(z) &= \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(|z|)} e^{p(\phi(|z|)-\phi(|z-re^{i\theta}|))} d\lambda(z) \\ &\le e^{p\phi(r)} \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(|z|)} d\lambda(z) \le e^{p\phi(r)} \|f\|_{p,w}^p. \end{aligned}$$

Applying this estimate to (15) then yields

$$\int_{\mathbb{C}} |D^n f(z)|^p e^{-p\phi(|z|)} d\lambda(z) \le \frac{n!^p}{2\pi r^{pn}} \int_0^{2\pi} e^{p\phi(r)} \|f\|_{p,w}^p d\theta = \frac{n!^p e^{p\phi(r)}}{r^p n} \|f\|_{p,w}^p.$$

Thus $\|D^n f\|_{p,w} \le n! r^{-n} e^{\phi(r)} \|f\|_{p,w}$, and it follows that $\|D^n\| \le n! r^{-n} e^{\phi(r)}$. This concludes the proof of the theorem. □

We also have the following result about the spectrum, $\sigma(D)$, of D. (See, e.g., [32] for a discussion of spectral theory and the functional calculus in this context.)

Theorem 9 ([2]) *Under the conditions of Theorem 8, the spectrum $\sigma(D)$ is given for any $a \geq 0$ by*

$$\sigma(D) = \Delta_a := \{z \in \mathbb{C} : |z| \leq a\}. \tag{16}$$

In particular, if $a = 0$, then $\Delta_a = \{0\}$.

Proof Let $e_\lambda(z) = e^{\lambda z}$ for $\lambda \in \mathbb{C}$. Clearly, we have $De_\lambda = \lambda e_\lambda$ and so e_λ is an eigenvector of the operator D with eigenvalue λ, as long as $e_\lambda \in B_w^p$. However, if $|\lambda| < a$ and we write $z = re^{i\theta}$ and $\lambda = |\lambda|e^{i\beta}$, we have the following:

$$|e_\lambda(z)e^{-\phi(|z|)}| = |e^{|\lambda|re^{i(\beta+\theta)}-\phi(r)}| = e^{|\lambda|r\cos(\beta+\theta)-\phi(r)} \leq e^{r\left(|\lambda|-\frac{\phi(r)}{r}\right)}. \tag{17}$$

But by (13), then this function is integrable, so $e_\lambda \in B_w^p$ for $|\lambda| < a$. Thus we have $\Delta_a \subseteq \sigma(D)$. To complete the proof we will show that $r(D)$, the spectral radius of D, satisfies the inequality $r(D) \leq a$. It suffices to show that $r(D) \leq a + \epsilon$ for any $\epsilon > 0$; so let us fix $\epsilon > 0$. Then again using (13), we see that there is $t_0 > 0$ such that $\phi(t) \leq (a + \epsilon)t$ for $t \geq t_0$. Thus by part 2 of Theorem 8 we have that $\|D^n\| \leq Cn!r^{-n}e^{(a+\epsilon)r}$ for any $r > 0$, $n = 1, 2, \ldots$, where C is a constant depending only on ϵ. Minimizing this expression with respect to r yields the critical value $r = \frac{n}{a+\epsilon}$. Substituting this choice of r gives $\|D^n\| \leq C\frac{n!e^n(a+\epsilon)^n}{n^n}$. Applying Stirling's formula gives that $\|D^n\| \leq f(n)$, where $f(n)$ is asymptotic to a constant times $\sqrt{n}(a + \epsilon)^n$. Thus we have $r(D) = \lim_{n\to\infty} \|D^n\|^{\frac{1}{n}} \leq a + \epsilon$. We conclude that $r(D) = a$ and $\sigma(D) = \Delta_a$, as desired. □

To further study this operator, we restrict our attention to the special case when $p = 2$, where we actually have a Hilbert space with inner product given by $(f, g) = \int_{\mathbb{R}^2} \bar{f}ge^{-2\phi(|z|)}d\lambda$. It is convenient to have a particular simple orthonormal basis to deal with, and, since we are dealing with entire functions that are guaranteed to have convergent power series, it makes sense to look at polynomials to try to find this orthonormal basis. It turns out that all we need are monomials.

Theorem 10 ([2]) *There exist constants c_n such that $\{u_n(z)\}$, where $u_n(z) = c_n z^n$, for $n \in \mathbb{N}_0$, forms an orthonormal basis for B_w^2.*

Proof First note that $(z^n, z^m) = 0$ if $n \neq m$. This follows from a simple calculation using the fact that the weight function is radial, so that using polar coordinates, we have

$$\begin{aligned}(z^n, z^m) &= \int_0^{2\pi} \int_0^\infty r^n e^{-in\theta} r^m e^{im\theta} e^{-2\phi(r)} r dr d\theta \\ &= \int_0^{2\pi} e^{i(m-n)\theta} d\theta \int_0^\infty r^{n+m+1} e^{-2\phi(r)} dr \\ &= 0\end{aligned}$$

if $n \neq m$. Note that the integral on r converges for any $n, m \in \mathbb{N}_0$ by the properties of our weight function. Thus the monomials form an orthogonal set. This orthogonal set is complete because every entire function has a convergent power series on $\mathbb{C}$. Thus, if we choose $c_n = \frac{1}{\|z^n\|}$, we normalize our set and, hence, the resulting sequence $\{u_n\}$ is an orthonormal basis for B_w^2. □

Now we specialize further by choosing the family of weight functions given by $w(z) = e^{-|z|^\alpha}$ for $\alpha \in \mathbb{R}$ with $0 < \alpha \leq 1$. (Note that in the notation of (13) and of Theorem 9 above, we then have $a = 1$ if $\alpha = 1$ and $a = 0$ if $0 < \alpha < 1$.) We will call the resulting Hilbert space $H_\alpha := B_w^2$. In this case, we can actually find the constants c_n explicitly.

Theorem 11 ([2]) *If $0 < \alpha \leq 1$, then for $n \in \mathbb{N}_0$, we have that*

$$\|z^n\|_{H_\alpha}^2 = \frac{2\pi}{\alpha} 2^{-\frac{2}{\alpha}(n+1)} \Gamma\left[\frac{2}{\alpha}(n+1)\right].$$

Proof Computing the norm in H_α gives

$$\|z^n\|_{H_\alpha}^2 = \int_{\mathbb{C}} |z|^n e^{-2r^\alpha} dz = \int_0^{2\pi} 1 d\theta \int_0^\infty r^{2n+1} e^{-2r^\alpha} dr.$$

For the integral over r, we make the change of variable $x = 2r^\alpha$, which changes the integral into

$$\|z^n\|_{H_\alpha}^2 = 2\pi \int_0^\infty \left(\left(\frac{x}{2}\right)^{\frac{1}{\alpha}}\right)^{2n+1} e^{-x} \frac{1}{2\alpha(\frac{x}{2})^{\frac{\alpha-1}{\alpha}}} dx = \frac{2\pi}{\alpha} 2^{-\frac{2}{\alpha}(n+1)} \int_0^\infty x^{\frac{2(n+1)}{\alpha}-1} e^{-x} dx.$$

However, the final integral is simply $\Gamma(\frac{2}{\alpha}(n+1))$. □

Thus we can simply take the normalizing constants c_n to be the square root of the reciprocal of the formula for $\|z^n\|^2$ given above. Namely,

$$c_n = \left(\frac{2\pi}{\alpha}\right)^{-\frac{1}{2}} 2^{\frac{n+1}{\alpha}} \left(\Gamma\left(\frac{2}{\alpha}(n+1)\right)\right)^{-\frac{1}{2}}$$

for every $n \in \mathbb{N}_0$. Now examining the action of D on a typical basis element $u_n(z)$, we see that: $Du_n = D(c_n z^n) = nc_n z^{n-1} = \frac{nc_n}{c_{n-1}} u_{n-1}$. We thus obtain the following representation of D:

Theorem 12 *The operator D is isomorphic to a weighted backward shift on H_α taking a sequence of coefficients (a_n) in $l^2(\mathbb{C})$, where $f(z) = \sum_{n=0}^\infty a_n u_n(z)$, to $(\gamma_n a_{n+1})$ in $l^2(\mathbb{C})$, where for $n \in \mathbb{N}_0$, $\gamma_n > 0$ and γ_n is given by $\gamma_n^2 = 2^{\frac{2}{\alpha}} \frac{(n+1)^2 \Gamma(\frac{2}{\alpha}(n+1))}{\Gamma(\frac{2}{\alpha}(n+2))}$.*

Proof Using the last calculation and writing $f(z) = \sum_{n=0}^{\infty} a_n u_n$, we obtain

$$Df(z) = \sum_{n=0}^{\infty} a_n \frac{nc_n}{c_{n-1}} u_{n-1} = \sum_{n=0}^{\infty} \frac{(n+1)c_{n+1}}{c_n} a_{n+1} u_n = \sum_{n=0}^{\infty} \gamma_n a_{n+1} u_n,$$

where $\gamma_n = \frac{(n+1)c_{n+1}}{c_n}$. It follows, using the previously calculated formula for c_n, that

$$\gamma_n^2 = \frac{(n+1)^2 c_{n+1}^2}{c_n^2} = \frac{(n+1)^2 \|z^n\|^2}{\|z^{n+1}\|^2} = 2^{\frac{2}{\alpha}} \frac{(n+1)^2 \Gamma(\frac{2}{\alpha}(n+1))}{\Gamma(\frac{2}{\alpha}(n+2))},$$

as desired. □

The last fact we will need from [2] is to apply the standard asymptotic for the Gamma function to obtain that $\gamma_n \sim c \cdot n^{1-\frac{1}{\alpha}}$ as $n \to \infty$, where c is a positive constant. Thus if $0 < \alpha < 1$, then $\gamma_n \to 0$ as $n \to \infty$.

Continuing beyond the results from [2], we start by calculating the adjoint D^*.

Theorem 13 *Given $f \in H_\alpha$, let $f = \sum_{n=0}^{\infty} a_n u_n$ be its expansion in terms of the orthonormal basis. The adjoint of D^* is isomorphic to a weighted forward shift given by the equation $D^*(a_n) = (\gamma_{n-1} a_{n-1})$.*

Proof To calculate D^*, write $D^* f = \sum_{n=0}^{\infty} b_n u_n$. Since $\{u_n\}$ is an orthonormal basis we find the nth, for $n \geq 1$, coefficient of $D^* f$: $(D^* f, u_n) = (f, Du_n) = (f, \gamma_{n-1} u_{n-1}) = \gamma_{n-1} a_{n-1}$. Thus we have $b_n = \gamma_{n-1} a_{n-1}$ for each $n \geq 1$. For b_0, we calculate $(D^* f, u_0) = (f, Du_0) = (f, 0) = 0$. Thus, D^* acts on the sequence of coefficients (a_n) as a weighted forward shift $(a_n) \mapsto (\gamma_{n-1} a_{n-1})$, with the new 0th coefficient being 0. □

Now that we have the adjoint, we can immediately see that D is not self-adjoint as D is a backward shift and D^* is a forward shift. Moreover, the following calculation with $f(z) \equiv 1$ shows that it is not even normal: Indeed, $D^* Df = D^* 0 = 0$, but, on the other hand, $DD^* f = D(\gamma_0 z) = \gamma_0$. This shows that we cannot apply the functional calculus for unbounded normal operators that was used in [20–23]. Instead we use the Riesz functional calculus, which is valid for bounded operators like D.

Next, we will use the asymptotic $\gamma_n \sim c \cdot n^{1-\frac{1}{\alpha}}$ to determine which trace ideals D will belong to, depending on α.

Theorem 14 *The operator D is compact on H_α for any $0 < \alpha < 1$, trace class for any $0 < \alpha < \frac{1}{2}$, Hilbert–Schmidt for any $0 < \alpha < \frac{2}{3}$, and, in general, $D \in J_p$ if $\alpha < \frac{p}{p+1}$ for any $p \in \mathbb{N}$, where J_p is the trace ideal defined in the previous section.*

Proof Let $0 < \alpha < 1$, and let $E_N : H_\alpha \to H_\alpha$ that takes a power series $\sum_{n=1}^{\infty} a_n u_n \mapsto \sum_{n=1}^{N} a_n \lambda_{n-1} u_{n-1}$, which is the composition DP_N of the derivative operator D with the projection onto the subspace of polynomials at most degree N, P_N. Each E_N is of finite rank, in fact, the range of E_N has dimension N. We claim that the norm limit of E_N is $D = \frac{d}{dz}$. Note that $\|D - E_N\| = \sup_{n>N}\{\lambda_{n-1}\}$.

But $\lambda_n \sim c \cdot n^{1-\frac{1}{\alpha}} \to 0$, as $n \to \infty$, for any $0 < \alpha < 1$. Thus E_N converges to D in norm and therefore D is compact. Furthermore, we can write $D = \sum_{n=1}^{\infty} \lambda_{n-1}(u_n, \cdot)u_{n-1}$ so λ_{n-1}^* are the singular values of D. To determine when λ_{n-1}^* are in l^p, we use the Limit Comparison Test to compare $\sum_{n=1}^{\infty}(\lambda_{n-1}^*)^p$ with $\sum_{n=1}^{\infty}(n^{1-\frac{1}{\alpha}})^p$, which converges if and only if $p\left(1-\frac{1}{\alpha}\right) < -1$. Solving this gives $\alpha < \frac{p}{p+1}$. Therefore $D \in J_p$ if $\alpha < \frac{p}{p+1}$ and, in particular, is trace class if $p < \frac{1}{2}$ and Hilbert–Schmidt if $p < \frac{2}{3}$. (Here, we have used the notation of Definition 5.) □

From now on, we will fix an α with $0 < \alpha < \frac{1}{2}$ and simply refer to H_α as H. In this case, we have the following spectrum for D.

Theorem 15 *We have $\sigma(D) = \sigma_p(D) = \{0\}$ and 0 is a simple eigenvalue of D with eigenfunction $f(z) \equiv 1$, the constant function equal to 1.*

Proof We know from Theorem 9 that $\sigma(D) = \Delta_a$, where $a = \lim_{t\to\infty} \frac{\phi(t)}{t}$, as in (13). Here we have $\phi(t) = t^\alpha$ for $0 < \alpha < \frac{1}{2}$. Thus we have that $\lim_{t\to\infty} \frac{t^\alpha}{t} = \lim_{t\to\infty} t^{\alpha-1} = 0$. It follows that $a = 0$ and hence, by Theorem 9, $\sigma(D) = \Delta_0 = \{0\}$. However, we also know that $f(z) \equiv 1 \in H$, so that D has the eigenvector f corresponding to the eigenvalue 0 and the point spectrum of D is also $\sigma_p(D) = \{0\}$. □

Finally, we turn to considering the set of operators $\{e^{-sD}\}_{s\in\mathbb{C}}$. We compare this to the result for ∂_c obtained in [23] and mentioned in Sect. 1. This theorem will show that D is the *infinitesimal shift* (of the complex plane).

Theorem 16 *The family $\{e^{-sD}\}_{s\in\mathbb{C}}$ gives the group of translation operators on H.*

Proof First note that since any $f \in H$ is an entire function, we have the convergent power series representation: $f(z-s) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!}(-s)^n$ for any $z, s \in \mathbb{C}$. Thus

$$e^{-sD}f(z) = \sum_{n=0}^{\infty} \frac{1}{n!}(-sD)^n f(z) = \sum_{n=0}^{\infty} \frac{1}{n!}(-s)^n \frac{d^n}{dz^n} f(z) = f(z-s).$$

This shows that e^{-sD} just acts as translation by s on the space H. From this expression we also see that $\lim_{s\to 0}\|e^{-sD}f - f\| = 0$. □

4 The Construction

With the results about D in hand, we turn to constructing an operator that might play the role of Frobenius when dealing with the Riemann zeta function or other entire or meromorphic functions of interest in number theory, analysis, or mathematical physics.

We begin by considering a particular choice of the family of analytic functions $\phi_\tau(z) = z + \tau$. This gives us operators

$$D_\tau := \phi_\tau(D) = D + \tau I, \tag{18}$$

for which the following lemma holds. (Recall that α has been fixed once for all to satisfy $0 < \alpha < \frac{1}{2}$ and hence, that Theorem 15 applies.)

Lemma 2 *For any $\tau \in \mathbb{C}$, $D_\tau \in B(H)$ with spectrum $\sigma(D_\tau) = \{\tau\}$. If $\tau \neq 0$, then D_τ is invertible and $D_\tau^{-1} \in B(H)$.*

Proof Applying the functional calculus on bounded operators along with the Spectral Mapping Theorem to the operator D and the function $\phi_\tau(z)$ gives a bounded operator D_τ with spectrum $\sigma(D_\tau) = \phi_\tau(\{0\}) = \{\tau\}$, where we have used Theorem 15 according to which $\sigma(D) = \{0\}$. Furthermore, if $\tau \neq 0$, then $0 \notin \sigma(D_\tau)$ and it follows that D_τ has a bounded inverse. □

This gives us a family of operators, each of whose spectra are each a single point, which can be any complex number. Recall that in the situation of the cohomology theory that helped prove the Weil conjectures, we would like an operator whose eigenvalues on different cohomology spaces are the zeros and poles of the zeta function we are interested in. In order to obtain an operator whose spectrum can represent the zero or pole set of a meromorphic function, we use the following construction. If $Z = \{z_1, z_2, \ldots\}$ is a (finite or countable) multiset of complex numbers, let H_n be a copy of the weighted Bergman space H and associate an operator D_n to be D_{z_n} on H_n. (Here and thereafter, a multiset is a set with integer multiplicities.) Finally, define the Hilbert space $H_Z = \bigoplus_n H_n$ with operator $D_Z = \bigoplus_n D_n$. This gives

Theorem 17 *For $Z = \{z_1, z_2, \ldots\}$, the operator D_Z, constructed above, has spectrum $\sigma(D_Z) = \overline{\{z_1, z_2, \ldots\}}$. Furthermore, for each $i \in \mathbb{N}$, z_i is an eigenvalue of D_Z and the number of linearly independent eigenvectors of z_i for D_Z in H_Z is equal to the number of times z_i occurs in the multiset $Z = \{z_1, z_2, \ldots\}$.*

Proof For each $n \in \mathbb{N}$, let $e_n \in H_Z$ be the element which is the constant function, with value 1 in the nth component, and 0 in every other component. Then $D_Z e_n = z_n e_n$ and so z_n is an eigenvalue with eigenvector e_n. Suppose $z_{n_1} = z_{n_2} = \cdots = z_{n_k} = z$. Then z is an eigenvalue with eigenvectors $e_{n_1}, e_{n_2}, \ldots e_{n_k}$ and so there are at least as many linearly independent eigenvectors of z for D_Z as the multiplicity of z in the multiset. Next, recall that the only eigenvalue of $\frac{d}{dz}$ on H is 0. Thus, the only eigenvalue of D_{z_n} is z_n. Suppose now that $D_Z x = zx$ for some x, we must either have the nth component of x being 0 or $z = z_n$ and so there cannot be any more linearly independent eigenvectors of z for D_Z. Now that we know z_i is an eigenvalue of D_Z for each i, we know that $\overline{\{z_1, z_2, \ldots\}} \subset \sigma(D_Z)$. Next, let $\lambda \in \mathbb{C} - \overline{\{z_1, z_2, \ldots\}}$. Then

$d = \inf_{n\geq 0} |\lambda - z_n| > 0$. Since $D = \frac{d}{dz}$ is quasinilpotent, $r(D) = 0$ and so there is a positive integer N such that for every integer $k \geq N$, we have $\|D^k\| < \left(\frac{d}{2}\right)^k$. Then, on the nth component of H_Z, we have that $\sum_{k=0}^{\infty} \frac{D^k}{|\lambda - z_n|}$ is absolutely convergent because

$$\sum_{k=N}^{\infty} \frac{\|D^k\|}{|\lambda - z_n|} \leq \sum_{k=N}^{\infty} \frac{\left(\frac{d}{2}\right)^k}{d^k} = \frac{1}{2^{N-1}}.$$

Then we can calculate the inverse on the nth component via the absolutely convergent series:

$$(\lambda I - D_{z_n})^{-1} = ((\lambda - z_n)I - D)^{-1} = \frac{1}{\lambda - z_n} \sum_{k=0}^{\infty} \frac{D^k}{\lambda - z_n}.$$

Further, by the same estimate $\|(\lambda I - D_{z_n})^{-1}\| \leq C$ uniformly in n, where

$$C = \sum_{k=0}^{N} \frac{\|D^k\|}{d^k} + 2^{1-N}. \tag{19}$$

Therefore, $\bigoplus_n (\lambda I - D_{z_n})^{-1} \in B(H_Z)$ and so $(\lambda I - D_Z)^{-1}$ exists and is bounded. That is, $\lambda \in \rho(D_Z)$, the resolvent set of D_Z; recall that by definition, $\sigma(D_Z)$ is the complement of $\rho(D_Z)$ in $\mathbb{C}$. Hence, $\sigma(D_Z) = \overline{\{z_1, z_2, \ldots\}}$. □

Corollary 1 *If $Z = \{z_1, z_2, \ldots\}$ is either the zero or pole set of a meromorphic function $f(z)$, then Z is a discrete set and so we have exactly $\sigma(D_Z) = Z$ counting multiplicity. Moreover, each z_i in Z is an eigenvalue of D_Z, with multiplicity equal to the multiplicity of z_i in the multiset Z.*

Thus, D_Z has all of the information from the multiset $\{z_1, z_2, \ldots\}$ contained in its spectrum. If we then consider the multiset to be the zeros and poles of a meromorphic function $f(z)$, then the operator D_Z contains these pieces of information of this function. This was, in fact, the original goal of this direction. In [23] (also [20, 21], and [22]) the spectrum of the operators studied there were vertical lines in the complex plane and the values of $\zeta(s)$ on these lines. This work was meant to approach the problem in a similar, but different, way and isolate out the zeros and poles of certain meromorphic functions. Theorem 17 gives a positive result that we have created such an operator. However, we wanted to go further and find a way to use determinant formulas to fully recover all of the values of the desired function as was done in the Weil conjectures. Unfortunately, we cannot use the determinant formulas for operators given in Sect. 2 for the operator D_Z to recover $f(z)$ as a whole, because with this formulation D_Z is not trace class. Even looking at just a single one of the terms, $D_\tau = D + \tau I$, with $\tau \neq 0$, is not compact, let alone trace class (or, consequently, in any of the operator ideals J_p), because our Hilbert space is infinite-dimensional. (Indeed, in a Banach space, the identity operator, like the closed unit ball, is compact if and only if the space is finite-dimensional.)

The first such modification we will make is to restrict each D_τ to its eigenspace, E, the space of constant functions. This restriction is then a compact operator. We also need to make a second adjustment from the original idea. Any zero or pole set of a meromorphic function will be a discrete set, and hence if there are infinitely many zeros or poles, they must tend to ∞. This would then imply that the operator D_Z described here is unbounded. What allows us to repair this problem and simultaneously recover the given function $f(z)$ using determinants, is to have our set Z consist of the reciprocals of the zeros rather than of the zeros themselves, and similarly for the poles. (In each case, the multiplicities of the zeros or the poles are taken into account.) The set of reciprocals will not necessarily be discrete as if the set is infinite, the sequence will tend to 0. However, rather than being a problem, this is actually completely required. Indeed, a compact operator on an infinite-dimensional space cannot have a bounded inverse and so 0 must be in the spectrum of any such compact operator. Combining this observation with the fact that regularized determinants apply only to trace ideals of compact operators, we see that having 0 in the closure of the set of reciprocals is necessary to apply the determinant theory to D_Z.

One comment to make about the restriction of the operator D_Z to its total eigenspace, E_Z, is that it simplifies the operator to a multiplication operator because the derivative on constant functions is just the zero operator. This is unfortunate as we do lose some of the rich setting of the Bergman space that has been used thus far. We are currently exploring alternative constructions in [5] that would allow us to remove this restriction and work on all of H. However, as we will see in the next section, the new version of the operator obtained by restriction will retain the desired spectrum from Theorem 17.

In addition, we observe that in some sense, by analogy with what happens for curves over finite fields for Weil-type cohomologies and with what is expected in more general situations associated with the Riemann zeta function and other L-functions (see, e.g., [10, 12] and [25, esp. Appendix B]), E_Z (the total eigenspace of D_Z) is the counterpart in our context of the total cohomology space (or, in the terminology of [25, 28], the total "fractal cohomology space") and correspondingly, the restriction of the original (generalized) Polya–Hilbert operator D_Z to its eigenspace E_Z is the counterpart of the linear endomorphism induced by the Frobenius morphism on the (total) cohomology space. (See Sect. 1.1 above.) Therefore, this modification of the original operator seems natural (and perhaps unavoidable) in order to obtain a suitable determinant formula, of the type obtained in Sects. 4.1 and 5 below.

4.1 Refining the Operator of a Meromorphic Function

First, let $Z = (z_n)$ be a sequence of complex numbers. Let $D_n := D + z_n I$ be the operator in the previous section restricted to the subspace of constant functions E. (It is clear that D_n is normal.) Let $D_Z = \bigoplus_n D_n$ act on the space $E_Z = \bigoplus_n E$ which

is a closed subspace of the Hilbert space H_Z from the last section. So in actuality, this new definition of D_Z is just the restriction to the Hilbert space E_Z of the operator given in the previous section. First we note that this restriction still retains the main property from the last section.

Theorem 18 *For each $n \in \mathbb{N}$, each z_n is an eigenvalue of D_Z and the number of linearly independent eigenvectors associated with z_n is equal to the number of times z_n occurs in the sequence Z. Furthermore, $\sigma(D_Z) = \overline{\{z_n : n = 1, 2, 3, \ldots\}}$.*

Proof Let e_n be the eigenfunctions from the previous proof. Then since e_n is constant in each coordinate, $e_n \in E_Z$. Thus when restricted to the space of functions constant on each coordinate, D_Z retains all of its eigenvalues and eigenvectors from before. Finally, we note that $\sigma(D_{z_n}) = \{z_n\}$ from which it follows that $\sigma(D_Z) = \overline{\{z_n : n = 1, 2, 3, \ldots\}}$ as in the proof of Theorem 17. □

Remark 1 Note that in the case when $Z = (z_n)$ is the sequence of the reciprocals of the nonzero elements in the zero set or the pole set of a given meromorphic function (as in Sect. 5 below), then $\sigma(D_Z) = Z$ if Z is finite and $\sigma(D_Z) = Z \bigcup \{0\}$ if Z is infinite.

The next theorem shows that this restriction of the operator will truly give us what we need for our quantized number theory framework.

Theorem 19 *We have the following relationships between an infinite sequence $Z = (z_n)$ and the associated operator D_Z.*

1) D_Z is bounded iff (z_n) is a bounded sequence.
2) D_Z is self-adjoint iff $z_n \in \mathbb{R}$ for all n.
3) D_Z is compact iff $\lim_{n\to\infty} z_n = 0$.
4) D_Z is Hilbert–Schmidt iff $\sum_{n=1}^{\infty} |z_n|^2 < \infty$.
5) D_Z is trace class iff $\sum_{n=1}^{\infty} |z_n| < \infty$.
6) For $p \geq 1$, $D_Z \in J_p$ iff $\sum_{n=1}^{\infty} |z_n|^p < \infty$.

If (z_n) is a finite sequence, then D_Z is bounded, compact, and in J_p for each $p \geq 1$.

Proof Since $\|D_n\| = |z_n|$, for each $n \in \mathbb{N}$, we have $\|D_Z\| = \sup_n |z_n|$. Then D_Z is bounded iff (z_n) is a bounded sequence. For 2), consider the sequence of operators $\overline{D}_N = \bigoplus_{n=1}^{N} D_n$, for $N \in \mathbb{N}$, as an operator on E_Z by letting it act as multiplication by 0 on the remaining components. Thus $\overline{D}_N$ is a finite rank operator on E_Z for each N. Then $\|D_Z - \overline{D}_N\| = \sup_{k>N} |z_k|$ and so if $\lim_{n\to\infty} z_n = 0$, we then have that D_Z is the norm limit of finite rank operators and thus is compact. On the other hand, if $\lim_{n\to\infty} z_n \neq 0$, then $\{e_n\}$ is a bounded sequence of vectors such that $\{D_Z e_n\}$ has no convergent subsequence. Thus, D_Z is not compact. For 3) and 4), assume that D_Z is compact. Then since $D_Z e_n = z_n e_n$ and the fact that $\{e_n\}$ forms an orthonormal basis for E_Z we know the singular values of E_Z are $\{z_n^*\}$, which is just the sequence of numbers $|z_n|$ arranged in nonincreasing order. Thus E_Z is Hilbert–Schmidt iff $\sum_{n=1}^{\infty} |z_n|^2 < \infty$, and, trace class iff $\sum_{n=1}^{\infty} |z_n| < \infty$ and, more generally, in J_p iff $\sum_{n=1}^{\infty} |z_n|^p < \infty$. Finally, if (z_n) is a finite sequence, then D_Z is actually a finite rank operator and is trivially bounded, compact, and in J_p for each $p \geq 1$. □

Note that the above result is well known from the theory of multiplication operators on sequence spaces and is just translated here in our setting. Now that we have a formulation that can indeed give us a trace class operator, we can state the result we will use to fully recover certain functions of interest.

Corollary 2 *If $\{z_n\}$ is a sequence of complex numbers satisfying $\sum_{n=1}^{\infty} |z_n| < \infty$, then we have* $\det(I - zD_Z) = \prod_n (1 - z_n z)$.

Proof This is a direct consequence of Eq. (8) for trace class operators of which D_Z is one when the series is absolutely summable. □

By the previous corollary, we can now see that we will be getting an entire function out of our construction. Thus if we want to handle meromorphic functions, we will need to handle zeros and poles separately. Also, we will want to choose our sequence (z_n) to be the reciprocals of the zeros and, separately, of the poles. With this in mind, we make the following final construction for our operator associated with a meromorphic function.

First, let $f(z)$ be an entire function on $\mathbb{C}$ with $z = 0$ not a zero of f. Let $\{a_n\}$ be a sequence of the zeros of $f(z)$, counting multiplicity. Define the sequence $Z = (z_n)$ where $z_n = \frac{1}{a_n}$. Define D_Z as before and call this D_f. Now given an integer $m \geq 1$, if we have $D_f \in J_m \setminus J_{m-1}$, then $\det_m(I - zD_Z)$ is well defined, where the regularized determinant $\det_m$ was defined in Sect. 2. (See, especially, Definition 6 and Theorem 5.) Finally, we note that if we are dealing with a meromorphic function instead of an entire function, we follow the lead from the proof of the Weil conjectures to simply take the ratio of these regularized determinants, with possibly the operator associated with the numerator being in different trace classes (that is, in different operator ideals) than that of the denominator.

In the next section, we will examine what this construction accomplishes for several classically important functions in number theory.

5 Applications of the Construction

In this section, we apply our construction to some special meromorphic functions of interest and conclude with showing that this construction does indeed give a suitable replacement for the Frobenius for any entire function of finite order.

5.1 Rational Functions

To begin, we look at the simplest type of meromorphic functions: the rational functions. Let $f(z)$ be a rational function. Then, we can write $f(z) = z^k g(z)$ for some $k \in \mathbb{Z}$ and further $g(z) = g(0)\dfrac{h(z)}{k(z)}$, where $h(z) = \displaystyle\prod_{n=1}^{s} \left(1 - \frac{z}{a_n}\right)$ and

$k(z) = \prod_{n=1}^{t}\left(1 - \frac{z}{b_n}\right)$ for some finite set $\{a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_t\}$. Construct the operator $D_{h(z)}$ and $D_{k(z)}$ as given in the previous chapter. The following theorem tells us that the determinant exactly recovers the given function f.

Theorem 20 *If $f(z)$ is a rational function as above, then*

$$f(z) = z^k g(0)\frac{\det_1(I - zD_{h(z)})}{\det_1(I - zD_{k(z)})}. \tag{20}$$

Proof Write out $f(z)$ as given in the preceding paragraph. Then consider the finite sequences $Z = \{\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_s}\}$ and $P = \{\frac{1}{b_1}, \frac{1}{b_2}, \ldots, \frac{1}{b_t}\}$. The operators D_Z and D_P are both trivially trace class since both are created from finite sequences. Hence, we may apply the 1-regularized determinant (really, just the normal Fredholm determinant since both operators are trace class) we defined to obtain

$$\frac{\det_1(I - zD_{h(z)})}{\det_1(I - zD_{k(z)})} = \frac{\prod_{n=1}^{s}(1 - \frac{z}{a_n})}{\prod_{n=1}^{t}(1 - \frac{z}{b_n})} = \frac{g(z)}{g(0)} \tag{21}$$

and thus, $f(z) = z^k g(0)\frac{\det_1(I - zD_{h(z)})}{\det_1(I - zD_{k(z)})}$. □

We consider $D_{h(z)}$ to be the analog of Frobenius for $h(z)$ (zeros) whereas $D_{k(z)}$ would be the analog for $k(z)$ (poles). Then this ratio of determinants would be considered a graded determinant associated with the Frobenius of the divisor (zeros minus poles) of $g(z)$.

5.2 Zeta Functions of Curves over Finite Fields

Recall that the zeta function of a (smooth, algebraic) curve Y over the finite field $\mathbb{F}_q$ is defined as $\zeta_Y(s) = \exp\left(\sum_{n=1}^{\infty}\frac{Y_n}{n}q^{-ns}\right)$. The proof of the Weil conjectures expressed this function as an alternating product of determinants as follows:

$$\zeta_Y(s) = \frac{\det(I - F^* q^{-s}|H^1)}{\det(I - F^* q^{-s}|H^0)\det(I - F^* q^{-s}|H^2)}.$$

One of the Weil conjectures, that $\zeta_Y(s)$ is a rational function of q^{-s}, then followed from this formula. Thus we may apply the result in the previous section about rational functions to obtain the following theorem.

Theorem 21 *Let Y be a smooth, projective, geometrically connected curve over $\mathbb{F}_q$, the field with q elements. Write $\zeta_Y(s) = \frac{f(q^{-s})}{g(q^{-s})}$ with $f(z), g(z)$ both polynomials. Then*

$$\zeta_Y(s) = \frac{\det_1\left(I - q^{-s}D_f\right)}{\det_1\left(I - q^{-s}D_g\right)}. \tag{22}$$

Proof We have that $\frac{f(z)}{g(z)}$ is a rational function of z. Thus by the rational function result: $\frac{f(z)}{g(z)} = \frac{\det_1(I-zD_{f(z)})}{\det_1(I-zD_{g(z)})}$ and so replacing with $z = q^{-s}$ gives: $\zeta_Y(s) = \frac{\det_1(I-q^{-s}D_f)}{\det_1(I-q^{-s}D_g)}$, as desired. □

Note that the results in this subsection can be extended in a straightforward manner to the zeta function of a (smooth, algebraic) d-dimensional variety over $\mathbb{F}_q$, where the integer $d \geq 1$ is arbitrary.

5.3 The Gamma Function

The next meromorphic function that we will turn our attention to is the Gamma function, defined initially by $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$ for $\Re(z) > 1$. This function has numerous applications in many branches of mathematics, including our focus—number theory. One point of interest is that this function gives a meromorphic continuation to all of $\mathbb{C}$ of the factorial function on integers. It also appears in the functional equation for the Riemann zeta function. We have the following well-known properties of the Gamma function:

Theorem 22 *For $z \in \mathbb{C}$, $z \notin \{0, -1, -2, \ldots\}$, we have*

1) $\Gamma(z+1) = z\Gamma(z)$.
2) $\Gamma(n) = (n-1)!$ *for* $n \in \mathbb{N}$.
3) $\Gamma(z) = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(\left(1+\frac{z}{n}\right)^{-1}e^{\frac{z}{n}}\right)$.

This infinite product representation for $\Gamma(z)$ allows us to now show that we can recover the function from the determinant of the operator construction we have laid out.

Theorem 23 *We have that for all $z \in \mathbb{C}$,*

$$\Gamma(z) = \frac{e^{-\gamma z}}{z}\frac{1}{\det_2(I - zD_{z\Gamma(z)})}. \tag{23}$$

Proof We will apply our construction to the function $g(z) = \frac{1}{z\Gamma(z)}$. This function is entire and has a simple zero at each negative integer. Note that the residue of $\Gamma(z)$ at $z = 0$ is 1; so that $g(0) = 1$. Now if we consider the sequence, $Z = (-\frac{1}{n})$, of reciprocals of zeros of $g(z)$, we see that it is not a summable series but that it is square summable. This means the associated operator D_Z is not trace class, but only Hilbert–Schmidt. This forces us to use $\det_2$ in our definition of the regularized determinant. In fact,

$$\det_2(I - zD_Z) = \prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right]. \tag{24}$$

This then leads to the following computation:

$$\begin{aligned}\frac{1}{\det_2(I-zD_{z\Gamma(z)})} &= \det_2(I-zD_Z)^{-1} \\ &= \left(\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right]\right)^{-1} \\ &= \prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)^{-1}e^{\frac{z}{n}}\right] \\ &= ze^{\gamma z}\Gamma(z).\end{aligned}$$

Therefore, we conclude that $\Gamma(z)$ is given by Eq. (23), as desired. □

5.4 The Riemann Zeta Function

Next, we turn our attention to another important example, the Riemann zeta function. First, we will consider the well-known Euler product expression for $\zeta(s)$.

Theorem 24 *For $s \in \mathbb{C}$, with $\Re(s) > 1$,*

$$\zeta(s) = \prod_p (1-p^{-s})^{-1},$$

where the product is taken over all prime numbers p.

To use our formulation, let $\phi(z) = 1-z$. Then, by the result for rational functions in Sect. 5.1 (Theorem 20), we have that $\phi(z) = \det(I-zD_\phi)$, which is true for every value of $z \neq 1$. Letting $z = p^{-s}$ then gives $(1-p^{-s})^{-1} = (\det(I-p^{-s}D_\phi))^{-1}$ for $s \neq \frac{2\pi ik}{\log p}$, $k \in \mathbb{Z}$. This leads to the following operator based Euler product:

Theorem 25 *For $s \in \mathbb{C}$, with $\Re(s) > 1$,*

$$\zeta(s) = \prod_p (\det(I-p^{-s}D_\phi))^{-1}, \tag{25}$$

where the product is taken over the primes p.

Proof We simply apply the determinant equality to each term in the infinite product and then use the standard Euler product convergence. Note that for $\Re(s) > 1$, we never have $s = \frac{2\pi ik}{\log p}$ for any integer k; so that the determinant equality does apply at each prime p. □

The completed zeta function, $\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}}s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s)$, is an entire function whose zeros all lie in the critical strip $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$ and coincide with the critical zeros of $\zeta(s)$. We have the following well-known product representation for $\xi(s)$.

Theorem 26 *For $s \in \mathbb{C}$ with $\Re(s) > 1$,*

$$\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}} e^{\left(\log(2\pi)-1-\frac{\gamma}{2}\right)s} \prod_{\rho} \left[\left(1-\frac{s}{\rho}\right) e^{\frac{s}{\rho}}\right], \tag{26}$$

where γ denotes Euler's constant and the infinite product over ρ is taken over all of the zeros of $\xi(s)$, which are the nontrivial (or critical) zeros of $\zeta(s)$.

Now if we wish to express $\xi(s)$ by using the determinant construction in this paper, we need to consider $Z = \{\frac{1}{\rho}\}$ and the convergence of $\sum_{\rho} \frac{1}{\rho^p}$. It is proven in [13] that this series converges for $p = 1$, but only conditionally and so we will need $p = 2$ to get the absolute convergence needed for $D_Z \in J_2$. Thus we must consider the determinant $\det_2(I - sD_{\xi(s)})$. This suggests the following theorem.

Theorem 27 *For all $s \in \mathbb{C}$,*

$$\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}} e^{\left(\log(2\pi)-1-\frac{\gamma}{2}\right)s} \det_2(I - sD_{\xi}). \tag{27}$$

Proof From the preceding discussion, we begin by defining $Z = \{\frac{1}{\rho}\}$, and constructing $D_{\xi(s)} = D_Z$. Then we calculate

$$\begin{aligned} \det_2(I - sD_{\xi(s)}) &= \det_2(I - sD_Z) \\ &= \prod_{\rho} \left[\left(1-\frac{s}{\rho}\right) e^{\frac{s}{\rho}}\right] \\ &= \frac{2\xi(s)}{\pi^{-\frac{s}{2}} e^{\left(\log(2\pi)-1-\frac{\gamma}{2}\right)s}}. \end{aligned}$$

Thus, $\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}} e^{\left(\log(2\pi)-1-\frac{\gamma}{2}\right)s} \det_2(I - sD_{\xi})$, as desired. This result is first obtained for $\Re(s) > 1$, and then upon analytic continuation, for all $s \in \mathbb{C}$. Indeed, both $\xi(s)$ and the regularized determinant define entire functions of s. □

We can then combine the results for $\xi(s)$ and $\Gamma(s)$ to give an expression for $\zeta(s)$ in a similar spirit to the representation of zeta functions of curves over finite fields, as follows.

Theorem 28 *For all $s \in \mathbb{C}$,*

$$\zeta(s) = -\frac{e^{(\log(2\pi)-1)s}}{2} \frac{\det_2(I - \frac{s}{2}D_{\frac{s}{2}\Gamma(\frac{s}{2})}) \det_2(I - sD_{\xi})}{\det_1(I - sD_{\phi})}, \tag{28}$$

where $\det_2(I-\frac{s}{2}D_{\frac{s}{2}\Gamma(\frac{s}{2})})$ gives the trivial zeros of $\zeta(s)$, $\det_2(I-sD_{\xi})$ gives the critical zeros of zeta, and $\det_1(I - sD_{\phi})$ gives the single pole at $s = 1$ with $\phi(s) := 1 - s$.

Proof We first recall the following three equations (see, in particular, Theorems 23 and 27), valid for all $s \in \mathbb{C}$:

$$\Gamma\left(\frac{s}{2}\right) = \frac{2e^{-\gamma\frac{s}{2}}}{s}\frac{1}{\det_2(I - \frac{s}{2}D_{\frac{s}{2}\Gamma(\frac{s}{2})})},$$

$$\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}}s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

and

$$\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}}e^{(\log(2\pi)-1-\frac{\gamma}{2})s}\det_2(I - sD_\xi).$$

We then solve for $\zeta(s)$ in the middle equation and substitute the other two to obtain successively:

$$\begin{aligned}\zeta(s) &= \frac{2\pi^{\frac{s}{2}}\xi(s)}{s(s-1)\Gamma(\frac{s}{2})}\\ &= 2\pi^{\frac{s}{2}}\cdot\frac{\frac{1}{2}\pi^{-\frac{s}{2}}e^{\log(2\pi)-1-\frac{\gamma}{2}s}\det_2(I - sD_\xi)}{s(s-1)\frac{2e^{-\gamma s}}{s}(\det_2(I - \frac{s}{2}D_{\frac{s}{2}\Gamma(\frac{s}{2})}))^{-1}}\\ &= \frac{e^{(\log(2\pi)-1)s}}{2}\frac{\det_2(I - \frac{s}{2}D_{\frac{s}{2}\Gamma(\frac{s}{2})})\det_2(I - sD_\xi)}{(s-1)}.\end{aligned}$$

Finally, letting $\phi(s) = 1 - s$ and using Theorem 20, since ϕ is a rational function, we can replace $s - 1 = -\det_1(I - sD_\phi)$ and obtain the desired equation. □

We will conclude this section with a different approach that gives an equivalent criterion for the Riemann Hypothesis. Let Z be the set of zeros of the function defined by $\hat{\xi}(s) = \xi\left(\frac{1}{2} + is\right)$. Construct the operator $D_Z = D_{\hat{\xi}}$. This leads to the following result.

Theorem 29 *The operator $D_{\hat{\xi}}$ is self-adjoint if and only if the Riemann hypothesis is true.*

Proof This follows directly from part (2) of Theorem 19 and the fact that the Riemann Hypothesis says that the zeros of $\xi\left(\frac{1}{2} + is\right)$ must all be real. □

It should be stressed that Theorem 29 does not as yet provide an approach to the Riemann hypothesis, for some of the reasons outlined in Sect. 6. We also note that Theorem 27 is potentially more useful than Theorem 28 (in part because it does not involve a determinant associated with the gamma function).

5.5 Hadamard's Factorization Theorem of Entire Functions

In this section, we observe that the theory presented here is quite general. It will apply to all entire functions of finite order. We will begin with an overview of the concepts of rank, genus, and order of an entire function as described in [7].

Definition 8 Let f be an entire function with zeros $\{a_1, a_2, \ldots\}$, repeated according to multiplicity and arranged such that $|a_1| \leq |a_2| \leq \cdots$. Then f is said to be of *finite rank* if there is an integer p such that $\sum_{n=1}^{\infty} |a_n|^{-(p+1)} < \infty$. If p is the smallest nonnegative integer such that this occurs, then f is said to be of *rank* p; a function with only a finite number of zeros has rank 0. A function is said to be of *infinite rank* if it is not of finite rank.

In order to define the genus of an entire function, we need to define what it means for an entire function to be written in standard form, which will require the following definition.

Definition 9 For $n \in \mathbb{N}_0$, define the *elementary factor*

$$E_n(z) = \begin{cases} (1-z), & \text{if } n = 0, \\ (1-z)\exp(\frac{z}{1} + \frac{z^2}{2} + \cdots \frac{z^n}{n}), & \text{if } n \geq 1. \end{cases}$$

To justify the definition of elementary factor, simply note that if $\sum_{n=1}^{\infty} |a_n|^{-(p+1)} < \infty$, then the infinite product $\prod_{n=1}^{\infty} E_p(\frac{z}{a_n})$ converges uniformly on compact subsets of $\mathbb{C}$ and defines an entire function with zeros at the complex numbers a_n, $n \geq 1$. The exponential factor is what is needed to ensure convergence of the infinite product. With this definition in hand, we can, in turn, define the genus of an entire function:

Definition 10 An entire function f has *finite genus* if the following statements hold: 1) f has finite rank p and 2) $f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right)$, where $g(z)$ is a polynomial of degree q. In this case, the *genus* of f is defined by $\mu = \max(p, q)$.

We now define the order of an entire function:

Definition 11 An entire function f is said to be of *finite order* if there is a positive constant a and $r_0 > 0$ such that $|f(z)| < \exp(|z|^a)$ for $|z| > r_0$. If f is not of finite order, then f is said to be of *infinite order*. If f is of finite order, then the number $\lambda = \inf\{a : |f(z)| < \exp(|z|^a)$ for $|z|$ sufficiently large$\}$ is called the *order* of f.

Thus the order of an entire function f is a measure of the growth of $|f(z)|$ as $|z| \to \infty$ whereas the rank of f is based on the growth of the nth smallest zero as $n \to \infty$. From the definitions, there is no inherent relationship between the two concepts, but with the following version of the Hadamard factorization theorem, we see that they are in fact closely related:

Theorem 30 (Hadamard Factorization Theorem) *If $f(z)$ is an entire function of finite order λ, then f has finite genus $\mu \leq \lambda$ and f admits the following factorization:*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right), \tag{29}$$

where $g(z)$ is a polynomial of degree $q \leq \lambda$ and $p = [\lambda]$, the integer part of λ. In particular, f is of rank not exceeding p.

Now when we apply our operator construction to a given entire function of finite order we obtain a Quantized Hadamard Factorization Theorem.

Theorem 31 (Quantized Hadamard Factorization Theorem) *If $f(z)$ is an entire function of finite order λ, then f admits the following factorization:*

$$f(z) = z^m e^{g(z)} \det{}_{p+1}(I - zD_{f(z)}), \tag{30}$$

where $g(z)$ is a polynomial of degree $q \leq \lambda$, and $p = [\lambda]$.

Proof By the standard Hadamard factorization theorem, we can write

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right), \tag{31}$$

where $g(z)$ is a polynomial of degree $q \leq \lambda$ and $p = [\lambda]$, with the rank of f not exceeding p. That is, if $\{a_1, a_2, \ldots\}$ is the multiset of zeros of $f(z)$ including multiplicity, then $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} < \infty$. Thus if $Z = \{\frac{1}{a_1}, \frac{1}{a_2}, \ldots\}$, the associated operator $D_Z \in J_{p+1}$. Then we can calculate successively:

$$\begin{aligned}
\det{}_{p+1}(I - zD_{f(z)}) &= \det{}_{p+1}(I - zD_Z) \\
&= \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{a_n}\right) \exp\left(\sum_{j=1}^{p} \frac{z^j}{j a_n^j} \right) \right] \\
&= \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right) \\
&= \frac{f(z)}{z^m e^{g(z)}}.
\end{aligned}$$

Thus we have that $f(z) = z^m e^{g(z)} \det_{p+1}(I - zD_{f(z)})$, as desired. □

In the above proof, we see that the extra convergence factor in the regularized determinants is exactly the same as the one for the elementary factor in the infinite product representation of entire functions, which validates, in some sense, the choice in this paper for the type of regularized determinants as those based on trace ideals. Thus the convergence factors needed in the usual Hadamard Factorization Theorem have an interpretation here relating to $\mathrm{Tr}(D^n)$.

6 Conclusion

We ended the previous section by giving what we called the Quantized Hadamard Factorization Theorem. This showed that the construction given in this paper can apply to any entire function of finite order and then, by taking ratios of determinants, can be extended to any meromorphic function which is a ratio of two such entire functions. This was worked out explicitly for the Riemann zeta function (see Theorems 27 and 28 above), and it has also been worked out by the authors for zeta functions of self-similar strings, both in the lattice and nonlattice case. (See [28, Chaps. 2 and 3] for background on self-similar fractal strings.) However, there was nothing in the construction preventing us from applying our results to even more general number-theoretic functions. In particular, a natural idea would be to try to extend our determinant formulas to other L-functions (see [33]). Could we then apply this construction to any zeta function (or, at least, to most zeta functions) from arithmetic geometry? This would require, essentially, knowledge about the existence of suitable meromorphic extensions of such functions, as well as about the asymptotic behavior of the zeros and poles of such extensions. Phrased differently, the L-functions for which our methods could be applied are those which can be suitably completed to become entire functions of finite order (or ratios of such entire functions). Furthermore, this naturally brings the consideration of the Selberg class of functions. See [33] or [25, Appendix E] (and the many references therein) for a discussion of these functions.

Another direction to take is to further justify why using ratios of these determinants is the correct method for handling meromorphic functions. In [25], the second author considers the properties of the Riemann zeta function as related to supersymmetric theory in physics and this ratio of determinants can be explained as a (regularized) Berezinian determinant from the theory of super linear algebra. However, this will not be further discussed here, but could be crucial for expanding upon the ideas presented in this work.

In this paper, we obtained a quantized version of the Hadamard factorization theorem, Theorem 31, but we expect to be able to generalize this result to obtain a quantized Weierstrass product formula; see [5].

With all of the successes obtained here, we must also admit the failures of this theory, at least in its present stage of its development. The construction of the operator for $\zeta(s)$ explicitly assumed knowledge of the zeros of $\zeta(s)$ and thus one could never hope to prove (RH) directly with this method. However, if we could find a different way to obtain the same function, by comparison you could extrapolate the zeros as was done with the Weil conjectures. That is, we need a suitable geometry and cohomology theory that would result in the same ratios of determinants of these operators. In the Weil conjectures, the geometry or points on the curve (over $\overline{\mathbb{F}_q}$, the algebraic closure of $\mathbb{F}_q$) corresponded to the fixed points of powers (or iterates) of the Frobenius operator. (Recall from our discussion just prior to Sect. 4.1 that in our context, the "fractal cohomology space" would seem to be the total eigenspace E_Z to which we restricted the original generalized Polya–Hilbert operator, viewed

as Frobenius acting on an appropriate analog of the underlying "curve.") Could we then consider the fixed points of the operator constructed in this paper? Analytically, this can be done by considering a suitable notion of generalized eigenfunctions (viewed as generalized tempered distributions). Thus far, however, this idea has not led to the development of a suitable working theory for the geometry underlying $\zeta(s)$. Providing an appropriate geometric framework is one of our main long-term objectives for future research on this subject.

Another interesting and related question (connected, in particular, to our discussion in Sects. 1.1 and 1.3) is whether the still conjectural fractal cohomology theory satisfies a suitable analogue of the Lefschetz fixed point formula (as stated in Theorem 1) for the counterpart of Frobenius.

One additional plan that we are currently working on is to rephrase the construction we have described here as a cohomology of sheaves in order to properly transition from the local setup given in this paper to a more global approach that might give new and useful information.

Acknowledgements The work of Michel L. Lapidus was supported by the US National Science Foundation (NSF) under the grant DMS-1107750.

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The Temptation of the Exceptional Characters

John B. Friedlander and Henryk Iwaniec

In celebration of forty years of collaboration

Abstract We survey some of the history and results related to the topic of the title with an emphasis admittedly biased toward our joint works thereon.

1 Introduction

Both on our own and together, the authors have been fascinated by the problems posed by the possible existence of exceptional Dirichlet characters and the profound influence these would play in some of the basic questions of multiplicative number theory. We take this opportunity to survey some of the history and results related to this topic, with an emphasis admittedly biased toward our joint works thereon.

2 The Class Number Formula

Let K be a number field and let D denote the absolute value of its discriminant. Recall that the Dedekind zeta-function of K

$$\zeta_K(s) = \sum_{\mathfrak{a}} (N\mathfrak{a})^{-s} = \frac{\kappa}{s-1} + O_K(1)$$

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H. Montgomery et al. (eds.), *Exploring the Riemann Zeta Function*,
DOI 10.1007/978-3-319-59969-4_4

has a simple pole at $s = 1$, with residue κ given in terms of some of the most important field constants by

$$\kappa = \frac{2^{r_1+r_2}\pi^{r_2}h(K)R}{\omega\sqrt{D}}$$

where h is the (wide) class number, R is the regulator, ω is the number of roots of unity, and r_1, r_2 are the number of real (respectively, complex conjugate pairs of) embeddings of K into the complex plane.

We shall throughout restrict ourselves to quadratic fields, so $K = \mathbb{Q}(\sqrt{\chi(-1)D})$ where $\chi = \chi_D$ is the real primitive Dirichlet character of conductor D. Now, the above constants simplify substantially and, by the Dirichlet class number formula, the residue is also given by

$$L(1, \chi) = \sum_{n\geqslant 1} \chi(n)n^{-1} .$$

We shall now denote the class number by $h(\pm D)$ rather than $h(K)$ and, since the class number is the order of a finite group, it follows that $L(1, \chi_D) > 0$, which we recall was a pivotal step in Dirichlet's proof [4]:

Theorem 1 *Let q be a positive integer and a an integer relatively prime to q. Then there are infinitely many prime numbers p with*

$$p \equiv a \ (\mathrm{mod} q) .$$

3 The Size of $L(1, \chi_D)$

It is important to have as much information as we can about the size of Dirichlet L-functions in general, especially at $s = 1$ and most especially for $\chi = \chi_D$. As far as upper bounds, we know that

$$L(1, \chi) \ll \log D$$

and that, under assumption of the Riemann Hypothesis for the L-function in question, we have

$$L(1, \chi) \ll \log\log D .$$

One would like to know more but the above is pretty good when compared to the situation for lower bounds where one knows, under the same RH, that

$$L(1, \chi) \gg (\log\log D)^{-1}$$

but what we know unconditionally is much less. From the class number formula it follows quickly that $L(1,\chi) \gg 1/\sqrt{D}$. Moreover, for even characters one improves this by a factor of $\log D$ by virtue of the presence of a non-trivial regulator.

4 The Zero-Free Region

It is important to have a good zero-free region for $L(s,\chi)$, $s = \sigma + it$, for all χ, but the real characters give extra problems (closely related to the size of $L(1,\chi)$). Specifically:

Theorem 2 (de la Vallée-Poussin, Landau [16]) *There exists a constant $c > 0$ such that*

$$L(s,\chi) \neq 0 \quad \text{for } \sigma > 1 - \frac{c}{\log(D(|t|+1))},$$

with the possible exception of a single real and simple zero $\beta = \beta_D$.

We call such D, χ, β "exceptional (with respect to the constant c)" and Landau proved that they are just that! More precisely:

Theorem 3 (Landau [16]) *For any given positive M there exists $c = c(M) > 0$ such that any given sequence of conductors D_n exceptional with respect to the constant c satisfies*

$$D_{n+1} > D_n^M$$

for all n.

Note that, in order for there to be any exceptional conductors, there must be an infinite family of them (since otherwise, knowing as we do that $L(1,\chi) > 0$, we could just choose c sufficiently small so as to rectify the finite number of obstructions to the inequality).

5 The Class Number

Theorem 4 (Hecke, Landau [16]) *As D runs through non-exceptional conductors, we have*

$$L(1,\chi) \gg \frac{1}{\log D}$$

and hence, for imaginary quadratic fields

$$h(-D) \gg \frac{\sqrt{D}}{\log D}\,.$$

A breakthrough idea came from Deuring [3].

Theorem 5 (Deuring [3], Mordell [19]) *If the Riemann zeta-function has any non-trivial zero off the line $\sigma = \frac{1}{2}$ then, for imaginary quadratic fields*

$$h(-D) \to \infty \quad \text{as } D \to \infty.$$

Deuring's idea was extended by Heilbronn [15] to give:

Theorem 6 (Heilbronn) *If any Dirichlet L-function has any non-trivial zero off the line $\sigma = \frac{1}{2}$ then, for imaginary quadratic fields*

$$h(-D) \to \infty \quad \text{as } D \to \infty.$$

Combining this last theorem with Hecke's result, [15] proves a famous conjecture of Gauss:

Corollary 1 (Heilbronn) *For imaginary quadratic fields*

$$h(-D) \to \infty \quad \text{as } D \to \infty.$$

Improving the quantitative estimate in Heilbronn's argument, Chowla became the first to deduce

Theorem 7 (Chowla [2]) *There are only a finite number of Euler's idoneal numbers (discriminants for which there is only one class in each genus).*

Indeed, he shows that the number of classes in the principal genus goes to infinity.

The first to beat the trivial class number bound by a fixed power was (again) Landau.

Theorem 8 (Landau [17]) *We have*

$$h(-D) \gg D^{\frac{1}{8}}/(\log D)^{\frac{3}{4}}\,.$$

Siegel sharpened that result to give what we know today, namely:

Theorem 9 (Siegel [20]) *For every ε and some $c_\varepsilon > 0$, we have*

$$L(1, \chi) > c_\varepsilon D^{-\varepsilon}\,.$$

Corollary 2 *We have*

$$h(-D) \gg D^{\frac{1}{2}-\varepsilon}\,.$$

Although, Siegel's paper does not mention zeros of L- functions, the theorem incidentally returns attention from zeros off the line to zeros near $s = 1$ since, by the mean-value theorem of differential calculus, one has the following result.

Corollary 3 *For every ε and some $c_\varepsilon > 0$, we have $L(\sigma, \chi) \neq 0$ for*

$$\sigma > 1 - c_\varepsilon D^{-\varepsilon} .$$

6 Primes in Arithmetic Progressions

Shortly after the proofs of the prime number theorem by Hadamard and de la Vallée-Poussin, the latter combined his techniques with those of Dirichlet to give an asymptotic formula for the number of primes up to a given point, in an arithmetic progression. We take $q \geqslant 1$, $(a, q) = 1$, and then have

$$\pi(x; q, a) = \sum_{\substack{p \leq x \\ p \equiv a (\bmod q)}} 1 \sim \frac{\pi(x)}{\varphi(q)}$$

and are interested in the range of uniformity in q for which this asymptotic formula can be shown to hold. There are three ranges for which this question is particularly noteworthy:

$$q \ll_\varepsilon x^{1-\varepsilon} ,$$

which is expected and would be a dream come true;

$$q \ll_\varepsilon x^{\frac{1}{2}-\varepsilon} ,$$

which follows from the assumption of RH for all L-functions with characters of modulus q:

$$q \ll_N (\log x)^N$$

which is known to hold for arbitrary N (the Siegel–Walfisz theorem).

An important related question is that of giving good upper bounds for $p_\ell = p_\ell(q, a)$, the least prime in the arithmetic progression in question. As immediate consequence of the above three ranges for the asymptotic formula one has

$$p_\ell \ll q^{1+\varepsilon}, \quad p_\ell \ll q^{2+\varepsilon}, \quad p_\ell \ll \exp(q^{1/N}) .$$

It was an achievement of the highest order when Linnik [18] gave an unconditional proof of a bound with a fixed power.

Theorem 10 *Let $q \geqslant 2$ and $(a, q) = 1$. There exists a positive universal constant L such that*

$$p_\ell(q, a) \ll q^L .$$

Many improvements have ensued in the known admissible values of "Linnik's constant" L.

It seems inevitable that separate proofs are required for the two different cases, depending on the existence or non-existence of exceptional zeros/characters. At first it seemed that the arguments in the exceptional case were the more difficult but more recently it has emerged that sometimes they can be simpler and stronger.

Thus, there is the following theorem of Heath-Brown [14].

Theorem 11 *Let $\varepsilon > 0$. Then, there exists $c(\varepsilon)$ such that if there is a zero $\beta \geqslant 1 - c(\varepsilon)/\log q$ then*

$$p_\ell(q, a) \ll q^{2+\varepsilon} .$$

What about other problems?

7 Twin Primes

Linnik's theorem offers an important example wherein the complete result can be proven, that is with or without the assumption of exceptional characters. It turns out, however, that there are very striking statements which can (up to the present) be proven only under the assumption of exceptional characters while proofs in the non-exceptional case remain a (probably remote) goal.

An example of this phenomenon was given by the following surprising theorem of Heath-Brown [13].

Theorem 12 *One (at least) of the following two alternatives must hold.*

1) There exists a positive constant c such that, for every q and every $\chi \mod q$ there are no real zeros β of $L(s, \chi)$ with

$$\beta > 1 - \frac{c}{\log q} ,$$

or

2) there are infinitely many prime pairs p, $p + 2$.

Remark 1 Recently, in [12] we have given a rather different account of this theorem.

8 More Primes in Arithmetic Progressions

For the remainder of the paper we are going to survey some of our joint works on the consequences of the assumption of the existence of exceptional characters. Here is a special case of our first work on this topic [7].

Theorem 13 *Let* $(a, q) = 1$, $r = 554,401$, *and* $\chi = \chi_D$. *For,*

$$D^r \leqslant q \leqslant \exp\left(L(1,\chi)^{-(r^r+1)^{-1}}\right),$$

we have

$$p_\ell(q,a) \ll q^{2-1/59}.$$

Actually, we get the asymptotic formula in this range. It is remarkable that one gets below the GRH exponent 2, but there are two important cautions.

1. The result only holds for intervals of q restricted in terms of D.
2. The interval is non-empty only under the "exceptional" assumption that

$$L(1,\chi_D) \ll (\log D)^{-r^r-1}.$$

9 Primes in Short Intervals

In [8] we proved the following result which, for short intervals, breaks the analogous Riemann Hypothesis barrier, albeit under constraints similar to those in the previous section.

Theorem 14 *Let* $\chi = \chi_D$, $x \geqslant D^r$ *with* $r = 18,290$ *and*

$$x^{\frac{1}{2}-\frac{1}{158}} < y \leqslant x.$$

Then

$$\pi(x) - \pi(x-y) = \int_{x-y}^{x} \frac{dt}{\log t}\left\{1 + O\left(L(1,\chi)(\log x)^{r^r}\right)\right\}.$$

Corollary 4 *If*

$$L(1,\chi) \ll (\log x)^{-r^r-1}$$

then

$$\pi(x) - \pi(x-y) = \int_{x-y}^{x} \frac{dt}{\log t}\left\{1 + O\left(\frac{1}{\log x}\right)\right\}.$$

It may seem strange to see the assumed upper bound for $L(1, \chi)$ phrased in terms of $\log x$ rather than $\log D$ but, of course, these are of the same order of magnitude when x is about a fixed power of D.

10 Prime Values of Polynomials

Recall that unconditionally we know [6] that there are infinitely many primes of the form $a^2 + b^4$. We hoped but did not succeed to prove that under the assumption of exceptional characters one might deduce an infinitude of primes of the form $a^2 + 1$. With a lot more work than in our other theorems of this type, we made a little progress. We state a couple of consequences of the results in [9].

Corollary 5 *Let* $\eta = \int_0^1 \sqrt{1 - t^6}dt$. *Suppose that*

$$L(1, \chi_D) \leqslant (\log D)^{-200} .$$

Then, for

$$\exp\big((\log D)^{10}\big) < x < \exp\big((\log D)^{16}\big)$$

we have

$$\sum_{a^2+b^6 \leqslant x} \Lambda\big(a^2 + b^6\big) = \frac{4}{\pi}\eta x^{2/3}\Big\{1 + O\left(\frac{1}{\log x}\right)\Big\}$$

so, if there are an infinity of such D, then there are an infinity of primes of the form $a^2 + b^6$.

Under a very similar assumption, one deduces the infinitude of elliptic curves over the rationals having prime discriminants, hence only a single prime of bad reduction.

11 Squares Plus Primes

Let $D \to \infty$ through fundamental discriminants (so, real quadratic fields). Let $\mathcal{N}(D)$ denote the number of solutions of

$$D = m^2 + p .$$

We write $\rho(p) = 1 + \chi_D(p)$ and note that this is just the number of solutions of the congruence $b^2 \equiv D (\mathrm{mod} p)$.

One conjectures that $\mathscr{N}(D) \sim H\sqrt{D}/\log D$ where

$$H = \prod_p \Big(1 - \frac{\rho(p)}{p}\Big)\left(1 - \frac{1}{p}\right)^{-1} .$$

Let us define

$$V(D) = \prod_{p<D}\left(1 - \frac{\rho(p)}{p}\right)$$

and

$$\ell(D) = L(1, \chi_D)\log D .$$

Theorem 15 (see [11]) *If $D > 0$ and $\ell(D) < 1$, then*

$$\mathscr{N}(D) = \sqrt{D}V(D)\left\{1 + O\left(\exp\left(-\frac{2}{5}\sqrt{\log 1/\ell(D)}\right)\right)\right\} .$$

Thus, we see:

Corollary 6 *i) If $\ell(D)$ is less than a sufficiently small positive constant c, then*

$$\mathscr{N} \asymp \sqrt{D}V(D)$$

ii) If $\ell(D) \to 0$ as $D \to \infty$, then

$$\mathscr{N} \sim \sqrt{D}V(D) .$$

Note that by Mertens' theorem we have $V(D) \sim e^{-\gamma}H/\log D$, so the last corollary is presumably giving us an answer that is wrong by the factor $e^{-\gamma}$. This is reminiscent of the wrong factor spit out by the sieve of Eratosthenes in a first attempt at counting primes (and for essentially the same reason).

12 Some Ideas About the Proofs

For a glimpse of the main idea involved one can argue as follows. Let χ be real and let us take $s = 1 + \varepsilon$ so that we have the Euler product

$$L(s, \chi) = \prod_p \big(1 - \chi(p)p^{-s}\big)^{-1}$$

and

$$\log L(s, \chi) = \sum_p \chi(p)p^{-s} + O_\varepsilon(1) .$$

Now, $L(s, \chi)$ is small if and only if $\log L(s, \chi)$ is large and negative and this can happen only if we have $\chi(p) = -1$ for most small primes p. Thus, on square-free integers χ should behave like the Möbius function μ. This reminds us of the sieve. But χ is periodic and, if we are looking near x which is huge compared to D, then the period is small, almost fixed, so our "sieve" is supercharged with very special (indeed, illusory) properties. On one hand, the character's periodicity enables harmonic analysis, on the other hand, its Möbius-like feature detects primes. The combination of these features brings tastes of the Hilbert–Polya dream.

In the next two sections, we shall sketch two different ways this simple idea can be utilized.

13 The Exceptional Conductor Influencing Itself

In this section we sketch the proof of the last stated theorem, that is we consider the equation $D = m^2 + p$.

We begin with the finite sequence

$$\mathscr{A} : n = D - m^2 \quad 0 < m < \sqrt{D}$$

and wish to estimate the sifted sum

$$S(\mathscr{A}, x) = \sum_{\substack{n \in \mathscr{A} \\ (n, P(x))=1}} 1$$

with $x = \sqrt{D}$ where $P(z)$ denotes the product of all primes $p < z$. We have

$$A_d = \sum_{\substack{n \in \mathscr{A} \\ n \equiv 0 (\bmod d)}} 1 = \frac{\rho(d)}{d} \sqrt{D} + O\big(\rho(d)\big)$$

where $\rho(d)$ denotes the number of solutions of the congruence $b^2 \equiv D (\mathrm{mod} d)$.

For square-free d,

$$\rho(d) \leqslant \lambda(d) = 1 * \chi(d)$$

where

$$\sum_{d \geqslant 1} \lambda(d) d^{-s} = \zeta(s) L(s, \chi_D) = \zeta_K(s) \, .$$

Note that $\lambda(d) = 0$ if there is any prime $p \mid d$ with $\chi(p) = -1$ so that, if χ is exceptional then usually $\rho(d) = \lambda(d) = 0$.

We next recall that by the fundamental lemma of sieve theory, we have

$$S(\mathscr{A}, z) = xV(z)\{1 + O(e^{-s})\} + \theta \sum_{d \leqslant y} \lambda(d) .$$

Here, the parameter y may be chosen arbitrarily subject to $y \geqslant z$,

$$|\theta| \leqslant 1 , \quad s = \frac{\log y}{\log z} \quad \text{and} \quad V(z) = \prod_{p<z} \Big(1 - \frac{\lambda(p)}{p}\Big) .$$

This gives a satisfactory count for the number of integers surviving the sieve up to a small range z but leaves a lot more to be done.

To proceed further we apply the Buchstab identity

$$S(\mathscr{A}, z) = S(\mathscr{A}, x) + \sum_{z \leqslant p < x} S(\mathscr{A}_p, p)$$

where $\mathscr{A}_p = \{\ell;\ \ell p \in \mathscr{A}\}$.

We need to bound the contribution from the sum on the right.

1. For primes p not awfully large, say those with $z \leqslant p \leqslant z^{-2}x$, we can apply an upper bound sieve. Although we are going to behave as though this sieve problem is usually of dimension zero, all we know for a fact is that its dimension is no larger than two. For these small primes this is good enough and applying a two-dimensional sieve we obtain

$$S(\mathscr{A}_p, p) \leqslant S(\mathscr{A}_p, z) \ll \lambda(p)\frac{x}{p}V(z) .$$

2. For the remaining primes, those with $z^{-2}x < p \leqslant x$, we use the trivial bound

$$S(\mathscr{A}_p, p) \leqslant |\mathscr{A}_p| \leqslant \lambda(p)\frac{x}{p} + \lambda(p) \leqslant 2\lambda(p)\frac{x}{p} .$$

In the normal course of affairs, this bound would ruin things, but not in our exceptional situation where we expect the sets $\mathscr{A}_p$ to be very small and $\lambda(p)$ to be zero for most p.

Combining the above bounds, we obtain

$$S(\mathscr{A}, z) - S(\mathscr{A}, x) \ll \delta(z)V(z)x + \delta(z^{-2}x)x$$

where

$$\delta(z) = \sum_{z \leqslant p \leqslant D} \frac{\lambda(p)}{p} .$$

Finally, we want to bound the function δ in terms of $L(1,\chi)$. It is convenient to use instead the truncated L-function

$$L^*(1,\chi) = \prod_{p>D}\Big(1-\frac{\chi(p)}{p}\Big)^{-1}$$

which is not so different and, in any case, satisfies the trivial bounds

$$L(1,\chi)(\log D)^{-1} \ll L^*(1,\chi) \ll L(1,\chi)\log D\,.$$

One can prove the following bound:

Proposition 1 *For* $0<\alpha\leqslant 1/17$ *we have*

$$\delta(D^\alpha) \ll \alpha^{-1}L^*(1,\chi)^{\alpha/2}\,.$$

All of these bounds are unconditional but we can deduce that $L(1,\chi)$ small implies $L^*(1,\chi)$ small which in turn implies that $\lambda(p)$ is small on average.

To quickly sketch the above proposition:

Step 1: By the Dirichlet hyperbola trick

$$\sum_{n\leqslant x}\lambda(n) = xL(1,\chi) + \text{error}\,.$$

Step 2: By partial summation, deduce that

$$\sum_{y<n\leqslant x}\lambda(n)n^{-1} = L(1,\chi)\log(x/y) + \text{error}\,.$$

Step 3: By the sieve,

$$\sum_{y<n\leqslant x,(n,P(z))=1}\lambda(n)n^{-1} \ll \prod_{p<z}\Big(1-\frac{1}{p}\Big)\Big(1-\frac{\chi(p)}{p}\Big)L(1,\chi)\log(x/y) + \text{error}\,.$$

The choices of $x = D$, y as a convenient small power of D and z as a still considerably smaller power of D yield the proposition.

14 Influence on the Nearby Conductors

In this section we consider a somewhat different line of reasoning whereby the exceptional conductor casts an effect, not on itself but on other conductors fairly far, but not too far, away in magnitude. This is the more common type of application of the principle and is already there in Landau's proof of the rarity of these objects.

We recall Linnik's combinatorial identity for primes. For $n \geqslant 2$, we have

$$\frac{\Lambda(n)}{\log n} = \sum_{k \geqslant 1} \frac{(-1)^{k-1}}{k} t_k(n)$$

where $t_k(n)$ is the number of ways of writing n as the product of k factors, each greater than one. After some technical difficulties (involving a preliminary sieving), this reduces the study of sums over primes

$$\sum_{n \leqslant x} a_n \frac{\Lambda(n)}{\log n} \quad \text{to} \quad \text{sums} \quad \sum_{n \leqslant x} a_n \tau_r(n)$$

for the ordinary divisor functions $r = 1, 2, \ldots$.

With this framework in mind, it makes sense to introduce some simpler identities which will operate well within our illusory backdrop. We have $\zeta_K(s) = \sum_{n \geqslant 1} \lambda(n) n^{-s}$ so that

$$\frac{1}{\zeta_K(s)} = \sum_m \nu(m) m^{-s} = \prod_p \Big(1 - \frac{1}{p^s}\Big)\Big(1 - \frac{\chi(p)}{p^s}\Big)$$

where $|\nu(m)| \leqslant \lambda(m) \leqslant \tau(m)$. Writing

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{L(s, \chi)\zeta'(s)}{\zeta_K(s)}$$

we infer that $\Lambda = \chi * \log * \nu = \lambda' * \nu$, say.

Here $\lambda' = \chi * \log$ and so, if χ has a "comparatively small" period, then λ' should behave like the ordinary divisor function τ.

Using this simple identity, we decompose the von Mangoldt function as

$$\Lambda(n) = \Lambda^*(n) + \Lambda_*(n) = \left(\sum_{m \leqslant D} + \sum_{m > D}\right) \sum_{dm = n} \lambda'(d)\nu(m) \, .$$

Here, Λ^* is easy to handle since the problematic factor ν, which hides the Möbius function, occurs with small m and λ' behaves like τ. On the other hand, Λ_* contributes to the error term in what appears at first glance to be unmanageable form. Specifically, if we use the simple bound

$$|\Lambda_*(n)| \leqslant \sum_{dm = n} \lambda'(d)\lambda(m) \, ,$$

then this behaves like the divisor function τ^2 or τ_4 and, for almost any sequence one can name, we have pretty limited knowledge of its distribution. One more elementary reduction is needed, a special case of Corollary 22.11 of [10].

Lemma 1 *Let $r \geqslant 2$, $\Delta = x^{1/r}$, $d \leqslant x$. Then*

$$\tau(d) \leqslant \sum_{\substack{\delta \mid d \\ \delta \leqslant \Delta}} \big(2\tau(\delta)\big)^A$$

with some $A \ll r \log r$.

Inputting the lemma, we see that, for $n \leq x$,

$$|\Lambda_*(n)| \leqslant (\log x) \sum_{\delta \leqslant \Delta} \big(2\tau(\delta)\big)^A \sum_{\delta \ell m = n} \lambda(m) \, .$$

Now, for each fixed δ we have replaced $\lambda' * \lambda$ by $1 * \lambda$ which is a function of type τ_3. In this way, the problem is reduced (of course, modulo technical details) to having to require knowledge of $\sum_{n \leqslant x} a_n \tau_k(n)$ only for $k = 1, 2, 3$.

We have sufficient knowledge of these arithmetic functions to sometimes go beyond the square-root barrier, that is $q > x^{1/2+\eta}$ in the case of arithmetic progressions due to [5], and $y < x^{1/2-\eta}$ in the case of short intervals, already much earlier due to Atkinson [1]. One may say that the crossing of this barrier is due to the power of harmonic analysis on $GL(3)$.

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Arthur's Truncated Eisenstein Series for $SL(2,Z)$ and the Riemann Zeta Function: A Survey

Dorian Goldfeld

Abstract Eisenstein series are not in $\mathcal{L}^2$. Maass, and later Selberg, naively truncated the constant term of Eisenstein series (for symmetric spaces of rank 1) so that the resulting truncated Eisenstein series was square integrable. This procedure was generalized to Eisenstein series on higher rank groups by Langlands and Arthur. In this survey we focus on the deep connections between Eisenstein series for $SL(2,\mathbf{Z})$, truncation, and the Riemann zeta function. Applications to zero free regions for the Riemann zeta function and automorphic L-functions are elucidated.

1 Introduction

Let $\mathfrak{h} := \{x + iy \mid x \in \mathbf{R}, y > 0\}$ denote the upper half plane. Then, as is well known, the modular group $\Gamma = SL(2,\mathbf{Z})$ acts on $\mathfrak{h}$ as follows. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $z \in \mathfrak{h}$, the action of γ on z is given by $\gamma z := (az+b)/(cz+d)$.

The Hilbert space $\mathcal{L}^2(\Gamma\backslash\mathfrak{h})$ of smooth $\mathcal{L}^2$ automorphic functions $f : \mathfrak{h} \to \mathbf{C}$ satisfying

$$f\left(\frac{az+b}{cz+d}\right) = f(z), \qquad \text{for all } z \in \mathfrak{h}, \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

with Petersson inner product

$$\langle f, g\rangle := \iint_{\Gamma\backslash\mathfrak{h}} f(z)\cdot\overline{g(z)}\ \frac{dxdy}{y^2}, \qquad f, g \in \mathcal{L}^2(\Gamma\backslash\mathfrak{h}),$$

has been extensively studied. Here $f \in \mathcal{L}^2$ means

$$\int_{\Gamma\backslash\mathfrak{h}} |f(x+iy)|^2\ \frac{dxdy}{y^2} < \infty,$$

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H. Montgomery et al. (eds.), *Exploring the Riemann Zeta Function*,
DOI 10.1007/978-3-319-59969-4_5

where $\Gamma\backslash\mathfrak{h}$ can be taken as the region $\mathcal{F} := \{z \in \mathfrak{h} \mid |z| \geq 1,\ |\Re(z)| \leq 1/2\}$ with congruent boundary points symmetric with respect to the imaginary axis.

One of the first examples of automorphic functions discovered is the Eisenstein series. For a complex variable s with $\Re(s) > 1$ and $z = x + iy \in \mathfrak{h}$, the Eisenstein series for the modular group $\Gamma = SL(2, \mathbf{Z})$ is defined to be

$$E(z, s) := \frac{1}{2} \sum_{\substack{c,\, d \in \mathbf{Z} \\ (c,d)=1}} \frac{y^s}{|cz + d|^{2s}}.$$

It is not hard to show that (see [9])

$$E\left(\frac{az + b}{cz + d}, s\right) = E(z, s), \qquad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

In this survey, we will examine the intimate connection between Eisenstein series and the Riemann zeta function $\zeta(s)$, which is already apparent in the well-known Fourier expansion [9]:

$$E(z, s) = y^s + \phi(s) y^{1-s} + \frac{2\pi^s \sqrt{y}}{\Gamma(s)\zeta(2s)} \sum_{n \neq 0} \sigma_{1-2s}(n) |n|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}\big(2\pi |n| y\big) e^{2\pi i n x}, \tag{1}$$

where

$$\phi(s) = \sqrt{\pi} \frac{\Gamma\left(s - \frac{1}{2}\right) \zeta(2s - 1)}{\Gamma(s)\zeta(2s)}, \quad \sigma_s(n) = \sum_{\substack{d|n \\ d \geq 1}} d^s,$$

$$K_s(y) = \frac{1}{2} \int_0^\infty e^{-\frac{y}{2}(u + u^{-1})} u^s \frac{du}{u}.$$

Here $K_s(y)$ is the modified Bessel function of the second kind.

Consider the completed Eisenstein series

$$E^*(z, s) := \zeta^*(2s) E(z, s),$$

where $\zeta^*(s) = \pi^{-\frac{s}{2}} \Gamma(s/2)\, \zeta(s) = \zeta^*(1 - s)$. Then $E^*(z, s)$ has the Fourier expansion

$$E^*(z, s) = \zeta^*(2s) y^s + \zeta^*(2 - 2s) y^{1-s} + 2\sqrt{y} \sum_{n \neq 0} \sigma_{1-2s}(n) |n|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |n| y) e^{2\pi i n x}$$

from which one easily deduces that

$$E^*(z, s) = E^*(z, 1 - s)$$

since $K_s(y) = K_{-s}(y)$ for any $s \in \mathbf{C}$ and $y > 0$. The analytic continuation of the Riemann zeta function then gives the meromorphic continuation of the Eisenstein series to the whole complex plane.

The modified Riemann hypothesis for a function $Z(s)$ asserts that all its zeros are either on the line $\Re(s) = 1/2$ or on the real axis in the interval $(0, 1)$. Hejhal [11] proved, in 1990, that the constant term

$$\zeta^*(2s)y^s + \zeta^*(2-2s)y^{1-s}$$

of the completed Eisenstein series satisfies the modified Riemann hypothesis if $y \geq 1$. His proof used the Maass–Selberg relation. This result was later extended in [13].

For fixed $s \in \mathbf{C}$, the Bessel function $K_s(y)$ has exponential decay in y for $y \to \infty$. This implies that for large $y > 0$, we have $E(z, s) \sim y^s + \phi(s)y^{1-s}$. Consequently

$$\int_{\Gamma\backslash\mathfrak{h}} \left|y^s + \phi(s)y^{1-s}\right|^2 \frac{dxdy}{y^2} > \int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=0}^{\infty} \left|y^s + \phi(s)y^{1-s}\right|^2 \frac{dxdy}{y^2} = \infty$$

for any value of $s \in \mathbf{C}$. It follows that the Eisenstein series is not in $\mathcal{L}^2(\Gamma\backslash\mathfrak{h})$. To overcome the fact that $E(z, s)$ is not in $\mathcal{L}^2$ and facilitate the computation of integrals involving Eisenstein series, Maass [16], in 1949, introduced the naive truncated Eisenstein series $E^T(z, s)$. Let $T > 0$. Then for $z \in \mathfrak{h}$, $s \in \mathbb{C}$,

$$E^T(z, s) := \begin{cases} E(z, s) & \text{if } y < T, \\ E(z, s) - y^s - \phi(s)y^{-s} & \text{if } y > T. \end{cases}$$

This function is not continuous in z but will lie in $\mathcal{L}^2$. It is also not automorphic, i.e.,

$$E^T\left(\frac{az+b}{cz+d},\ s\right) \neq E^T(z, s)$$

for most $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

In 1962, Selberg [19] extended Maass' work on naive truncated Eisenstein series to the more general situation of symmetric spaces of rank 1. The inner product of two truncated Eisenstein series (Maass–Selberg relations), which was based on Green's theorem, played a crucial role in the computation of the Selberg trace formula [18]. The more general Arthur–Selberg trace formula (see [15]) has been central for the advancement of modern number theory.

One of the most basic tools in the theory of automorphic forms on higher rank groups is the truncation operator which was first introduced by Langlands [14] in 1966. In 1978–1980 Arthur [1, 2] made an important advance by extending and making much more explicit Langlands' truncation operator, especially when applied to Eisenstein series. In particular, Arthur's truncation involves a summation $\sum_F (-1)^{\operatorname{codim}(F)} \mathcal{E}_F$ for a certain characteristic function $\mathcal{E}_F$ over the faces F of a certain convex polyhedron in a Euclidean space V.

The key property we want to point out for this survey is that Arthur's truncated Eisenstein series are automorphic which was not the case with Maass and Selberg's naively truncated Eisenstein series. Arthur's truncated Eisenstein series greatly simplified the computation of the Maass–Selberg relations, especially in the situation of higher rank groups. The use of differential operators (via Green's theorem) was avoided and all proofs of the Maass–Selberg relations were reduced to algebraic computations as in the Rankin–Selberg method [9].

In this survey we focus on Arthur's truncated Eisenstein series for the modular group Γ and its connections with the Riemann zeta function. The reader may find it useful to concurrently look at other surveys on truncation such as [4–6].

2 Arthur's Truncated Eisenstein Series

In order to define Arthur's truncated Eisenstein series for Γ we require some preliminary definitions. A function $F : \mathfrak{h} \to \mathbf{C}$ is said to be periodic if $F(z+1) = F(z)$ for all $z \in \mathfrak{h}$.

Definition (Constant Term) For a piecewise continuous periodic function $F : \mathfrak{h} \to \mathbf{C}$ define

$$CF(z) := \int_0^1 F(z+u)\, du, \qquad (\text{for } z \in \mathfrak{h}),$$

to be the constant term of F.

Definition (Truncated Constant Term) Let $T > 0$ and $z = x + iy \in \mathfrak{h}$. For an integrable periodic function $F : \mathfrak{h} \to \mathbf{C}$ define

$$C^T F(z) := \begin{cases} 0 & \text{if } 0 \le y \le T, \\ CF(z) & \text{if } y > T, \end{cases}$$

to be the truncated constant term of F.

Definition (Arthur's Truncation Operator) Let $T > 0$ and $z \in \mathfrak{h}$. For an integrable periodic function $F : \mathfrak{h} \to \mathbf{C}$ define

$$\Lambda^T F(z) := F(z) - \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} C^T F(\gamma z),$$

to be the truncation of the function $F(z)$ with respect to T. Here

$$\Gamma_\infty := \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \quad m \in \mathbf{Z} \right\}$$

is the stabilizer of ∞ under the action of Γ.

Definition (Polynomial Growth) A periodic function $\phi : \mathfrak{h} \to \mathbf{C}$ has polynomial growth if there exist constants $C, B > 0$ such that

$$|\phi(z)| \ \leq \ C \operatorname{Im}(z)^B, \qquad (\operatorname{Im}(z) \to \infty).$$

The following theorem summarizes the main properties of the truncation operator Λ^T.

Theorem 2.1 (Properties of Λ^T) *Let $T > 0$. The truncation operator Λ^T satisfies the following properties.*

- *Let $\phi \in \mathcal{L}^2(\Gamma\backslash\mathfrak{h})$ be a cusp form (the constant term of ϕ is zero) then $\Lambda^T\phi = \phi$.*
- *Let $\phi : \Gamma\backslash\mathfrak{h} \to \mathbf{C}$ be integrable. Assume $T \geq 1$ and $z \in \mathcal{F}$. Then $\Lambda^T\phi(z) = \phi - C^T\phi(z)$.*
- *Assume $\phi : \Gamma\backslash\mathfrak{h} \to \mathbf{C}$ is smooth, is an eigenfunction of the invariant Laplacian, and has polynomial growth. Then $\Lambda^T\phi \in \mathcal{L}^2(\Gamma\backslash\mathfrak{h})$.*
- *Let $\eta, \phi \in \mathcal{L}^2(\Gamma\backslash\mathfrak{h})$. Then $\langle \Lambda^T\eta, \ \phi\rangle = \langle \eta, \Lambda^T\phi\rangle$.*

Proof of Theorem 2.1 It is clear that if ϕ is a cusp form then truncation has no effect since the constant term of ϕ is zero. Next, assume z lies in the fundamental domain $\mathcal{F}$. Then for $\gamma \in \Gamma/\Gamma_\infty$, with $\gamma \notin \Gamma_\infty$, the complex number γz must lie outside of $\mathcal{F}$ and not in a translate $m + \mathcal{F}$ with $m \in \mathbf{Z}$. It is easy to see that in this situation, we must have

$$\operatorname{Im}(\gamma z) < 1.$$

The second bulleted item immediately follows from this. The third bulleted item is much more difficult to prove, so we do not give the details. The last bulleted item is easily proved with a change of variables $z \mapsto \gamma^{-1}z$ for $\gamma \in \Gamma/\Gamma_\infty$. □

We now carefully examine the action of the truncation operator on the Eisenstein series $E(z, s)$ with Fourier expansion given in (1). Then

$$CE(z, s) = y^s \ + \ \phi(s)y^{1-s},$$

and for $T > 0$,

$$C^T E(z, s) = \left(y^s \ + \ \phi(s)y^{1-s}\right) \cdot \mathfrak{char}_{[T,\infty]}(y),$$

where

$$\mathfrak{char}_{[\alpha,\beta]}(y) = \begin{cases} 1 & \text{if } \alpha \leq y \leq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

It immediately follows that for $T > 0$, Arthur's truncated Eisenstein series $\Lambda^T E(z,s)$ is given by

$$\Lambda^T E(z,s) = E(z,s) - \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma \\ \mathrm{Im}(\gamma z) \geq T}} \Big(\mathrm{Im}(\gamma z)^s + \phi(s)\, \mathrm{Im}(\gamma z)^{1-s} \Big). \tag{2}$$

3 Fourier Expansion of Arthur's Truncated Eisenstein Series

We now compute the Fourier expansion of $\Lambda^T E(z,s)$. The proof of the Fourier expansion requires the following lemma.

Lemma 3.1 *Let $v \in \mathbf{C}$ with $\Re(v) > 0$. Then*

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{Y^w}{w^{1+v}}\, dw = \begin{cases} \frac{(\log Y)^v}{\Gamma(1+v)} & \text{if } 1 \leq Y, \\ 0 & \text{if } 0 \leq Y < 1. \end{cases}$$

Proof of Lemma 3.1 For $\Re(w) > 0$ and $\Re(v) > 0$ we have

$$w^{-v} = \frac{1}{\Gamma(v)} \int_0^\infty e^{-wx} x^v\, \frac{dx}{x}.$$

It follows that

$$\begin{aligned} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{Y^w}{w^{1+v}}\, dw &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{Y^w}{w} \cdot \left(\frac{1}{\Gamma(v)} \int_0^\infty e^{-wx} x^v\, \frac{dx}{x} \right) dw \\ &= \frac{1}{\Gamma(v)} \int_0^\infty x^v \cdot \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \left(\frac{Y}{e^x} \right)^w \frac{dw}{w}\, \frac{dx}{x} \\ &= \frac{1}{\Gamma(v)} \int_0^{\log Y} x^v\, \frac{dx}{x} = \frac{(\log Y)^v}{\Gamma(1+v)}. \quad \square \end{aligned}$$

It immediately follows from (2) and Lemma 3.1 that

$$\Lambda^T E(z,s) = E(z,s) - E^T(z,s) - \phi(s) E^T(z, 1-s),$$

where we have defined

$$E^T(z,s) := \lim_{v \to 0} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{T^{-w}}{w^{1+v}} E(z, s+w)\, dw.$$

Warning: The function $E^T(z,s)$ is not the naive truncation of the Eisenstein series that we considered earlier in the introduction.

Theorem 3.2 (Fourier Expansion of E^T) *Let $T > 0$, and $x + iy \in \mathfrak{h}$. Let $\mu(n)$ denote the Mobius function. Then*

$$\int_0^1 E^T(x+iy,s)e^{-2\pi inx}\,dx$$

$$= \begin{cases} y^s \cdot \mathfrak{char}_{[T,\infty]}(y) \;+\; \frac{\mathfrak{char}_{[0,1/T]}(y)}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{T^{-w}}{w}\phi(s+w)\,y^{1-s-w}\,dw & \text{if } n=0, \\ 0 & \text{if } yT \geq 1 \text{ and } n \neq 0, \\ 2y^{1-s} \sum\limits_{\substack{d|n \\ yTd^2\ell^2 \leq 1 \\ d\geq 1}} \sum\limits_{\ell=1}^{\infty} d^{1-2s}\,\frac{\mu(\ell)}{\ell^{2s}} \int_0^{\sqrt{\frac{1}{Tyd^2\ell^2}-1}} \frac{\cos(2\pi nyt)}{(t^2+1)^s}\,dt & \text{otherwise.} \end{cases}$$

Theorem 3.3 (Constant Term of $\Lambda^T E(z,s)$) *Let $T > 0$ and $z = x + iy \in \mathfrak{h}$. Then the constant term of $\Lambda^T E(z,s)$ is given by*

$$\int_0^1 \Lambda^T E(z+u,s)\,du = \left(y^s + \phi(s)y^{1-s}\right)\cdot\left(1 - \mathfrak{char}_{[T,\infty]}(y)\right)$$
$$- \mathfrak{char}_{[0,1/T]}(y)\cdot\frac{1}{2\pi i}$$
$$\cdot \int_{2-i\infty}^{2+i\infty} \frac{T^{-w}}{w}\left(\phi(s+w)y^{1-s-w} \;+\; \phi(s)\phi(1-s+w)y^{-s+w}\right)dw.$$

Proof of Theorem 3.2 With the Fourier expansion (1), we compute

$$\int_0^1 E^T(x+iy,s)e^{-2\pi inx}\,dx$$

$$= \begin{cases} \lim\limits_{v\to 0} \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{T^{-w}}{w^{1+v}}\left(y^{s+w} + \phi(s+w)y^{1-s-w}\right)dw & n=0, \\ \lim\limits_{v\to 0} \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{T^{-w}}{w^{1+v}} \frac{2\pi^{s+w}\sqrt{y}}{\Gamma(s+w)\zeta(2s+2w)} \sum\limits_{n\neq 0}\sigma_{1-2s-2w}(n)|n|^{s+w-\frac{1}{2}}K_{s+w-\frac{1}{2}}(2\pi|n|y)\,dw & n\neq 0. \end{cases}$$

We evaluate the above integrals individually. It follows from Lemma 3.1 that

$$\lim_{v\to 0}\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{T^{-w}}{w^{1+v}}\left(y^{s+w}+\phi(s+w)y^{1-s-w}\right)dw$$
$$= y^s\cdot\mathfrak{char}_{[T,\infty]}(y) \;+\; \left(\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{T^{-w}}{w}\phi(s+w)\,y^{1-s-w}\,dw\right)\cdot\mathfrak{char}_{[0,1/T]}(y).$$

Note that if $yT > 1$, then the integral on the right above is easily seen to be zero by shifting the line of integration to the right. This is the reason why we can multiply the integral by $\mathfrak{char}_{[0,\,1/T]}(y)$. This establishes the first case of Theorem 3.2.

Next, we consider the integral involving the K-Bessel function. To simplify the computation we make use of the following integral representation:

$$K_{s+w-\frac{1}{2}}(2\pi|n|y) = \frac{\Gamma(s+w)\left(4\pi|n|y\right)^{s+w-\frac{1}{2}}}{\sqrt{\pi}} \int_0^\infty \frac{\cos(t)}{\left(t^2+4\pi^2n^2y^2\right)^{s+w}}\,dt.$$

It follows that

$$\lim_{v\to 0}\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{T^{-w}}{w^{1+v}}\frac{2\pi^{s+w}\sqrt{y}}{\Gamma(s+w)\zeta(2s+2w)}\sigma_{1-2s-2w}(n)|n|^{s+w-\frac{1}{2}}$$
$$K_{s+w-\frac{1}{2}}\left(2\pi|n|y\right)dw$$

$$= \lim_{v\to 0}\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{T^{-w}}{w^{1+v}}\frac{2^{2s+2w}\pi^{2s+2w-1}y^{s+w}}{\zeta(2s+2w)}\sum_{d|n}d^{1-2s-2w}\,|n|^{2s+2w-1}$$
$$\int_0^\infty\frac{\cos(t)}{\left(t^2+4\pi^2n^2y^2\right)^{s+w}}\,dt\,dw$$

$$= \lim_{v\to 0}\,2(2\pi|n|)^{2s-1}y^s\sum_{\ell=1}^\infty\frac{\mu(\ell)}{\ell^{2s}}\sum_{d|n}d^{1-2s}\int_{t=0}^\infty\frac{\cos(t)}{\left(t^2+4\pi^2n^2y^2\right)^s}$$
$$\cdot\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left(\frac{4\pi^2n^2y}{T\ell^2d^2\left(t^2+4\pi^2n^2y^2\right)}\right)^w\frac{dw}{w^{1+v}}\,dt.$$

Now,

$$\lim_{v\to 0}\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left(\frac{4\pi^2n^2y}{T\ell^2d^2\left(t^2+4\pi^2n^2y^2\right)}\right)^w\frac{dw}{w^{1+v}}$$
$$= \begin{cases}1 & \text{if } \frac{4\pi^2n^2y}{T\ell^2d^2\left(t^2+4\pi^2n^2y^2\right)} > 1,\\ 0 & \text{otherwise.}\end{cases}$$

The condition

$$\frac{2\pi^2n^2y}{T\ell^2d^2\left(t^2+4\pi^2n^2y^2\right)} > 1$$

is equivalent to

$$yT\ell^2 d^2 \le 1 \qquad \text{and} \qquad 0 \le t^2 \le 4\pi^2 n^2 y^2 \left(\frac{1}{yT\ell^2 d^2} - 1\right).$$

It follows that

$$\lim_{v\to 0} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{T^{-w}}{w^{1+v}} \frac{2\pi^{s+w}\sqrt{y}}{\Gamma(s+w)\zeta(2s+2w)} \sigma_{1-2s-2w}(n)|n|^{s+w-\frac{1}{2}} K_{s+w-\frac{1}{2}}(2\pi|n|y)\, dw$$

$$= 2(2\pi)^{2s-1} y^s \sum_{\substack{d|n \\ yT\ell^2 d^2 \le 1}} \sum_{\ell=1}^{\infty} |n|^{2s-1} \frac{\mu(\ell)}{\ell^{2s}} d^{1-2s} \int_{t=0}^{2\pi|n|y\sqrt{\frac{1}{yT\ell^2 d^2}-1}} \frac{\cos(t)}{\left(t^2 + 4\pi^2 n^2 y^2\right)^s}\, dt$$

$$= 2y^{1-s} \sum_{\substack{d|n \\ yTd^2\ell^2 \le 1}} \sum_{\ell=1}^{\infty} d^{1-2s} \frac{\mu(\ell)}{\ell^{2s}} \int_0^{\sqrt{\frac{1}{Tyd^2\ell^2}-1}} \frac{\cos(2\pi nyt)}{(t^2+1)^s}\, dt.$$

If $n \neq 0$ and $yT > 1$ it is not possible to have integers $d, \ell \ge 1$ such that $yTd^2\ell^2 \le 1$. Therefore the above integral vanishes and the second case of Theorem 3.2 has been proved. The third case of Theorem 3.2 follows immediately from the above. □

Remarks In [2], Arthur defined truncated Eisenstein series in great generality. The explicit computation of the Fourier expansion of Arthur's truncated Eisenstein series for higher rank groups, such as $SL(n, \mathbf{Z})$ with $n > 2$, seems out of reach at present. This is not the case for $SL(2, \mathbf{Z})$ because of the existence of so many special known integral representations for Bessel functions. Our knowledge of higher rank Whittaker functions is much more limited at present.

4 Maass–Selberg Relation for $SL(2, Z)$

In this section we compute the inner product of two truncated Eisenstein series which is called the Maass–Selberg relation. Arthur's proof of the Maass–Selberg relation just uses the Rankin–Selberg method [9]. It is an algebraic proof based on the fact that if a group G acts on a topological space X then the union of all translates

$g \cdot (G\backslash X)$ (with $g \in G$) is just the space X. More generally for a subgroup $H \leq G$, we also have

$$\bigcup_{g \in H\backslash G} g \cdot (G\backslash X) \;=\; H\backslash X.$$

Arthur's algebraic proof stands in stark contrast to the original analytic proof of Maass and Selberg which required differential operators and Green's theorem.

Theorem 4.1 (Maass–Selberg Relation) *Let $s, s' \in \mathbf{C}$ and $T \geq 1$. Then*

$$\left\langle \Lambda^T E(*,s),\ E(*,s') \right\rangle = \frac{T^{s+\overline{s'}-1}}{s+\overline{s'}-1} + \overline{\phi(s')}\frac{T^{s-\overline{s'}}}{s-\overline{s'}} + \phi(s)\frac{T^{\overline{s'}-s}}{\overline{s'}-s} + \phi(s)\overline{\phi(s')}\frac{T^{1-s-\overline{s'}}}{1-s-\overline{s'}}.$$

Proof of Theorem 4.1 It is clear that

$$\left\langle \Lambda^T E(*,s),\ \Lambda^T(E(*,s') \right\rangle$$
$$= \iint_{\Gamma\backslash\mathfrak{h}} \Lambda^T E(z,s) \cdot \left(\overline{E(z,s) \;-\; E^T(z,s) - \phi(s)E^T(z,1-s)} \right) \frac{dxdy}{y^2}. \tag{3}$$

To simplify the computation of (3) we require a lemma.

Lemma 4.2 *We have* $\left\langle \Lambda^T E(*,s),\ \Lambda^T(E(*,s') \right\rangle = \left\langle \Lambda^T E(*,s),\ (E(*,s') \right\rangle$.

Proof of Lemma 4.2 By (3), it is enough to prove that

$$\iint_{\Gamma\backslash\mathfrak{h}} \Lambda^T E(z,s) \cdot \left(\overline{E^T(z,s') + \phi(s')E^T(z,1-s')} \right) \frac{dxdy}{y^2} \;=\; 0.$$

This can be shown as follows:

$$\iint_{\Gamma\backslash\mathfrak{h}} \Lambda^T E(z,s) \cdot \left(\overline{E^T(z,s') + \phi(s')E^T(z,1-s')} \right) \frac{dxdy}{y^2}$$
$$= \iint_{\Gamma\backslash\mathfrak{h}} \Lambda^T E(z,s)$$
$$\sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \left(\overline{\left(\mathrm{Im}(\gamma z)^{s'} + \phi(s')\mathrm{Im}(\gamma z)^{1-s'} \right) \cdot \mathfrak{char}_{[T,\infty]}(\mathrm{Im}(\gamma z))} \right) \frac{dxdy}{y^2}$$
$$= \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \iint_{\gamma^{-1}(\Gamma\backslash\mathfrak{h})} \Lambda^T E(z,s) \left(\overline{y^{s'} + \phi(s')y^{1-s'} \cdot \mathfrak{char}_{[T,\infty]}(y)} \right) \frac{dxdy}{y^2}$$
$$= \iint_{\Gamma_\infty\backslash\mathfrak{h}} \Lambda^T E(z,s) \overline{\left(y^{s'} + \phi(s')y^{1-s'} \right)} \cdot \mathfrak{char}_{[T,\infty]}(y) \frac{dxdy}{y^2}$$
$$= \int_0^1 \int_0^\infty \Lambda^T E(z,s) \left(y^{\overline{s'}} + \overline{\phi(s')}y^{1-\overline{s'}} \right) \cdot \mathfrak{char}_{[T,\infty]}(y) \frac{dxdy}{y^2}.$$

This procedure is termed: "*unravelling* $\overline{E^T(z,s)+\phi(s)E^T(z,1-s)}$". The x-integral above picks off the constant term of $\Lambda^T E(z,s)$ which is given in Theorem 3.3 and equals

$$\left(y^s+\phi(s)y^{1-s}\right)\cdot\left(1-\mathfrak{char}_{[T,\infty]}(y)\right)-\mathfrak{char}_{[0,\,1/T]}(y)\cdot\frac{1}{2\pi i}$$
$$\cdot\int_{2-i\infty}^{2+i\infty}\frac{T^{-w}}{w}\left(\phi(s+w)y^{1-s-w}\ +\ \phi(s)\phi(1-s+w)y^{-s+w}\right)dw.$$

Since

$$\left(1-\mathfrak{char}_{[T,\infty]}(y)\right)\cdot\mathfrak{char}_{[T,\infty]}(y)\ =\mathfrak{char}_{[0,\,1/T]}(y)\cdot\mathfrak{char}_{[T,\infty]}(y)\ =\ 0$$

for $T\geq 1$ the proof of the lemma follows immediately. □

To complete the proof of the Maass–Selberg relation we apply Lemma 4.2 and unravel $\Lambda^T E(z,s)$. We assume for the moment that $\Re(s+s')>1$ and $\Re(s)>\Re(s')$. It follows that

$$\begin{aligned}
\left\langle\Lambda^T E(*,s),\ \Lambda^T(E(*,s')\right\rangle &= \left\langle\Lambda^T E(*,s),\ (E(*,s')\right\rangle\\
&= \int_0^1\int_0^\infty\left[y^s-\left(y^s+\phi(s)y^{1-s}\right)\mathfrak{char}_{[T,\infty]}(y)\right]\cdot\overline{E(z,s')}\,\frac{dxdy}{y^2}\\
&= \int_0^\infty\left[y^s-\left(y^s+\phi(s)y^{1-s}\right)\mathfrak{char}_{[T,\infty]}(y)\right]\cdot\left(y^{\overline{s'}}+\overline{\phi(s')}y^{1-\overline{s'}}\right)\,\frac{dy}{y^2}\\
&= \int_0^\infty y^s\left(y^{\overline{s'}}+\overline{\phi(s')}y^{1-\overline{s'}}\right)\,\left(1-\mathfrak{char}_{[T,\infty]}(y)\right)\frac{dy}{y^2}\\
&\quad-\int_0^\infty\phi(s)y^{1-s}\left(y^{\overline{s'}}+\overline{\phi(s')}y^{1-\overline{s'}}\right)\,\mathfrak{char}_{[T,\infty]}(y)\,\frac{dy}{y^2}\\
&= \int_0^T\left(y^{s+\overline{s'}-2}+\overline{\phi(s')}y^{s-\overline{s'}-1}\right)dy-\int_T^\infty\phi(s)\left(y^{-s+\overline{s'}-1}+\overline{\phi(s')}y^{-s-\overline{s'}}\right)dy\\
&= \frac{T^{s+\overline{s'}-1}}{s+\overline{s'}-1}+\overline{\phi(s')}\frac{T^{s-\overline{s'}}}{s-\overline{s'}}+\phi(s)\frac{T^{\overline{s'}-s}}{\overline{s'}-s}+\phi(s)\overline{\phi(s')}\frac{T^{1-s-\overline{s'}}}{1-s-\overline{s'}}.
\end{aligned}$$

This identity then extends by analytic continuation in s,s' to all $s,s'\in\mathbf{C}$.

□

5 A Zero Free Region for the Riemann Zeta Function

It was proved independently by Hadamard and de la Vallée Poussin in 1896 that the Riemann zeta function $\zeta(s)$ has no zeros on the line $\Re(s)=1$ (see [20]). Only a few different proofs of this are now known, but the most intriguing is the proof, first discovered by Selberg (unpublished), that is based on the Eisenstein series $E(z,s)$.

Recall the completed Eisenstein series

$$E^*(z,s) := \zeta^*(2s)E(z,s),$$

with $\zeta^*(s) = \pi^{-\frac{s}{2}}\Gamma(s/2)\zeta(s) = \zeta^*(1-s)$, where $E^*(z,s)$ has the Fourier expansion

$$E^*(z,s) = \zeta^*(2s)y^s + \zeta^*(2-2s)y^{1-s} + 2\sqrt{y}\sum_{n\neq 0}\sigma_{1-2s}(n)|n|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx}.$$

Now if $\zeta(1+it_0) = 0$ for some real number t_0 this implies that

$$E^*\big(z,\ (1+it_0)/2\big) = 2\sqrt{y}\sum_{n\neq 0}\sigma_{-it_0}(n)|n|^{it_0}K_{it_0/2}(2\pi|n|y)e^{2\pi inx}$$

which would be a (not identically zero) square integrable non-holomorphic cusp form for $SL(2,Z)$. This is not possible! A simple way to see that it is not possible is to consider the inner product $\big\langle E^*\big(\ *, (1+2it_0)/2\big),\ E^*\big(\ *, (1+2it_0)/2\big)\big\rangle$ which is a convergent integral (because of the rapid decay of the K-Bessel function) which must have a positive value > 0. This is a contradiction because by analytic continuation in the variable w it follows from the Rankin–Selberg unfolding method that

$$\big\langle E^*\big(\ *, (1+it_0)/2\big),\ E^*\big(\ *, \overline{w}\big)\big\rangle = \int_0^1\int_0^\infty E^*(z,\ (1+it_0)/2)\cdot\zeta^*(2w)\,y^w\,\frac{dxdy}{y^2} = 0$$

which vanishes for all $w \in \mathbf{C}$ because the constant term of $E^*(z,\ (1+it_0)/2)$ is zero.

This method was later generalized by Jacquet and Shalika [12] where they prove that all automorphic L-functions of the group $GL(n)$ do not vanish on the line $\Re(s) = 1$. The Eisenstein approach is very powerful. It is possible to obtain standard zero free regions of the form

$$\Re(s) > 1 - \frac{c}{\log(2+\mathrm{Im}(s))^B},$$

with explicit constants $B, c > 0$, for many families of L-functions by use of the Maass–Selberg relations following an approach developed by Sarnak [17] in 2004. In this regard see [3, 7, 8, 10]. All these recent results specifically require the theory of Arthur's truncated Eisenstein series for higher rank groups.

To complete this survey, we present a short sketch of Sarnak's 2004 method in the setting of Arthur's truncated Eisenstein series. The following theorem (proved in [17]) is equivalent to a standard zero free region for $\zeta(s)$.

Theorem 5.1 *There exists an effectively computable constant $C > 0$ such that*

$$|\zeta(1+2it)| \ > \ \frac{C}{\log\big(2+|t|\big)^3}$$

for all $t \in \mathbf{R}$.

Proof of Theorem 5.1 Let $T > 1$, $0 < \eta < 1$ be fixed and $t \in \mathbf{R}$ with $|t|$ large. The main idea of the proof is to evaluate the following integral:

$$\mathcal{I}^T(t,\eta) := |\zeta(1+2it)|^2 \int_\eta^\infty \int_0^1 \left|\Lambda^T E(z,\ 1/2+it)\right|^2 \ \frac{dxdy}{y^2}$$

in two different ways.

First computation of $\mathcal{I}^T(t,\eta)$ (Upper Bound):

We will prove the upper bound

$$\boxed{\mathcal{I}^T(t,\eta) \ \ll \ \frac{|\zeta(1+2it)}{\eta} \cdot \big(\log(2+|t|)^2 + 2\log(T)\big).} \tag{4}$$

To prove (4) note that

$$\mathcal{I}^T(t,\eta) = |\zeta(1+2it)|^2 \iint_{\Gamma\backslash\mathfrak{h}} N(z,\eta) \left|\Lambda^T E(z,\ 1/2+it)\right|^2 \ \frac{dxdy}{y^2}$$

where

$$N(z,\eta) := \#\big\{\gamma \in \Gamma_\infty\backslash\Gamma \ \big|\ \mathrm{Im}(\gamma z) > \eta\big\}.$$

Since $0 < \eta < 1$, we have the bound $N(z,\eta) \ll \frac{1}{\eta}$ which may be combined with the Maass–Selberg relation

$$\iint_{\Gamma\backslash\mathfrak{h}} \left|\Lambda^T E(z,\ 1/2+it)\right|^2 \ \frac{dxdy}{y^2} = 2\log(T) \ - \ \frac{\phi'}{\phi}(1/2+it) \ + \ \frac{\overline{\phi(1/2+it)}T^{2it} + \phi(1/2+it)T^{-2it}}{2it}.$$

This gives the upper bound (4) .

Second computation of $\mathcal{I}^T(t,\eta)$ (Lower Bound):

Assume $\eta = T^{-1}$. We will prove the lower bound

$$\boxed{\mathcal{I}^T(t,\eta) \ \gg \ \frac{1}{\eta} \cdot \frac{1}{\log(2+|t|)}.} \tag{5}$$

It follows from Theorem 3.2 that if $y \geq T^{-1}$ then

$$\int_0^1 E^T(x+iy)e^{-2\pi inx}\,dx = 0.$$

Recall that

$$\Lambda^T E(z,s) = E(z,s) \ - \ E^T(z,s) - \phi(s)E^T(z,1-s).$$

It follows that if $\eta = T^{-1}$ then the nonzero Fourier coefficients of $E(z,s)$ coincide with the nonzero Fourier coefficients of $\Lambda^T E(z,s)$ for $y \geq \eta$. Hence

$$\begin{aligned}\mathcal{I}^T(t,\eta) &\gg \sum_{m=1}^{\infty} |\sigma_{-2it}(m)|^2 \int_{\eta m}^{\infty} \left| \frac{K_{it}(2\pi y)}{\Gamma\left(\frac{1}{2}+it\right)} \right|^2 \frac{dy}{y} \\ &\gg \frac{1}{|t|} \sum_{m \leq \frac{|t|}{4\eta}} |\sigma_{-2it}(m)|^2 \\ &\gg\gg \frac{1}{|t|} \sum_{p \leq \frac{|t|}{4\eta}} |\sigma_{-2it}(p)|^2\end{aligned}$$

Using sieve theory one may show that $|\sigma_{-2it}(p)| \gg 1$ for a positive proportion of primes. We thus obtain the lower bound (5). There is a loss of one logarithm in the lower bound because (for large x) the number of primes $\leq x$ is asymptotic to $x/\log x$ by the prime number theorem.

Theorem 5.1 immediately follows by comparing the lower bound (5) with the upper bound (4). □

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On a Cubic Moment of Hardy's Function with a Shift

Aleksandar Ivić

Abstract An asymptotic formula for

$$\int_{T/2}^{T} Z^2(t)Z(t+U)dt \qquad (0 < U = U(T) \leqslant T^{1/2-\varepsilon})$$

is derived, where

$$Z(t) := \zeta\left(\tfrac{1}{2}+it\right)\left(\chi\left(\tfrac{1}{2}+it\right)\right)^{-1/2} \quad (t \in \mathbb{R}), \quad \zeta(s) = \chi(s)\zeta(1-s)$$

is Hardy's function. The cubic moment of $Z(t)$ is also discussed, and a mean value result is presented which supports the author's conjecture that

$$\int_{1}^{T} Z^3(t)dt \;=\; O_\varepsilon(T^{3/4+\varepsilon}).$$

1 Introduction

Let $\zeta(s)$ denote the Riemann zeta-function, and as usual let us define *Hardy's function* $Z(t)$ as

$$Z(t) := \zeta\left(\tfrac{1}{2}+it\right)\left(\chi\left(\tfrac{1}{2}+it\right)\right)^{-1/2} \quad (t \in \mathbb{R}), \quad \zeta(s) = \chi(s)\zeta(1-s).$$

We have (see, e.g., [5, Chap. 1] and [6]), with $s = \sigma + it$ a complex variable,

$$\chi(s) = 2^s\pi^{s-1}\sin\left(\tfrac{1}{2}\pi s\right)\Gamma(1-s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-1/2} e^{i(t+\pi/4)}\left(1+O\left(\frac{1}{t}\right)\right) \qquad (1)$$

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H. Montgomery et al. (eds.), *Exploring the Riemann Zeta Function*,
DOI 10.1007/978-3-319-59969-4_6

if $t \geqslant t_0$ (> 0). In fact, one can obtain a full asymptotic expansion for $\chi(s)$ from the classical Stirling formula for the gamma-function.

The function $Z(t)$ (see [8] for an extensive account) is real-valued, even, smooth, and as $|Z(t)| = |\zeta(\frac{1}{2} + it)|$, its zeros coincide with the zeros of $\zeta(s)$ on the "critical line" $\Re e\, s = \frac{1}{2}$. It is indispensable in the study of the zeros of $\zeta(s)$ on the critical line.

The moments of $Z(t)$ are of interest, but since $|Z(t)| = |\zeta(\frac{1}{2} + it)|$, only odd moments represent a novelty. Korolev [14] and Jutila [10, 11] showed independently that

$$\int_0^T Z(t)\,dt = O(T^{1/4}), \quad \int_0^T Z(t)\,dt = \Omega_{\pm}(T^{1/4}),$$

thus establishing (up to the value of the numerical constants which are involved) the true order of the integral in question. This improves on the author's bound $O_\varepsilon(T^{1/4+\varepsilon})$, obtained in [7].

However, higher odd moments of $Z(t)$, where one also expects a lot of cancellations due to the oscillatory nature of the function $Z(t)$, remain a mystery.

In [8, Eq. (11.9)], an explicit formula for the cubic moment of $Z(t)$ was presented. This is

$$\int_T^{2T} Z^3(t)\,dt = 2\pi\sqrt{\frac{2}{3}} \sum_{(\frac{T}{2\pi})^{3/2} \leqslant n \leqslant (\frac{T}{\pi})^{3/2}} d_3(n) n^{-\frac{1}{6}} \cos\left(3\pi n^{\frac{2}{3}} + \tfrac{1}{8}\pi\right) + O_\varepsilon(T^{3/4+\varepsilon}), \tag{2}$$

where as usual $d_3(n)$ is the divisor function

$$d_3(n) = \sum_{k\ell m = n} 1 \qquad (k, \ell, m, n \in \mathbb{N}),$$

generated by $\zeta^3(s)$ for $\Re e\, s > 1$.

Remark 1 Here and later ε denotes positive constants which are arbitrarily small, but are not necessarily the same ones at each occurrence, while $a(x) \ll_\varepsilon b(x)$ (same as $a(x) = O_\varepsilon(b(x))$) means that the $|a(x)| \leqslant Cb(x)$ for some $C = C(\varepsilon) > 0, x \geqslant x_0$. By $a \asymp b$ we mean that $a \ll b \ll a$. The symbol $f(x) = \Omega_{\pm}\big(g(x)\big)$ means that both $\limsup_{x\to\infty} f(x)/g(x) > 0$ and $\liminf_{x\to\infty} f(x)/g(x) < 0$ hold.

The motivation for this investigation is the following problem, first posed by the author in Oberwolfach 2003 during the conference "Elementary and Analytic Number Theory": Does there exist a constant c with $0 < c < 1$ such that

$$\int_1^T Z^3(t)\,dt \;=\; O(T^c)? \tag{3}$$

This is Eq. (11.8) of [8]. So far obtaining (3) with any $c < 1$ is an open problem, but if one considers the cubic moment of $|Z(t)|$, then it is known that

$$T(\log T)^{9/4} \ll \int_1^T |Z(t)|^3\,dt = \int_1^T |\zeta\left(\tfrac{1}{2}+it\right)|^3\,dt \ll T(\log T)^{9/4}, \tag{4}$$

which establishes the true order of the integral in question. However, obtaining an asymptotic formula for this integral remains a difficult problem. The lower bound in (4) follows from general results of Ramachandra (see his monograph [15]), and the upper bound is a recent result of Bettin et al. [2].

A strong conjecture of the author is that

$$\int_1^T Z^3(t)\,dt = O_\varepsilon(T^{3/4+\varepsilon}). \tag{5}$$

Note that (5) would follow from (2) and the bound

$$\sum_{N<n\leqslant N'\leqslant 2N} d_3(n)e^{3\pi i n^{2/3}} \ll_\varepsilon N^{2/3+\varepsilon}. \tag{6}$$

It may be remarked that the exponential sum in (6) is "pure" in the sense that the function in the exponential does not depend on any parameter as, for example, the sum

$$\sum_{N<n\leqslant N'\leqslant 2N} n^{it} = \sum_{N<n\leqslant N'\leqslant 2N} e^{it\log n} \qquad (1\leqslant N\ll \sqrt{t}),$$

which appears in the approximation to $\zeta(\frac{1}{2}+it)$ (see, e.g., Theorem 4.1 of [5]), depends on the parameter t. However, the difficulty in the estimation of the sum in (6) lies in the presence of the divisor function $d_3(n)$ which, in spite of its simple appearance, is quite difficult to deal with.

Remark 2 A natural question is to ask: What is the minimal value of c for which (3) holds ($c = 1+\varepsilon$ is trivial)? In other words, to try to find an omega result for the integral in (3). A related and interesting problem is to investigate integrals of $Z(t)$ with "shifts", i.e., integrals where one (or more) factor $Z(t)$ is replaced by $Z(t+U)$. The parameter U, which does not depend on the variable of integration t, is supposed to be positive and $o(T)$ as $T\to\infty$, where T is the order of the range of integration.

Some results on such integrals already exist in the literature. For example, Hall [3] proved that, for $U = \alpha/\log T, \alpha \ll 1$, we have uniformly

$$\int_0^T Z(t)Z(t+U)\,dt = \frac{\sin\alpha/2}{\alpha/2}T\log T + (2\gamma - 1 - 2\pi)T\cos\alpha/2 + O\left(\frac{\alpha T}{\log T} + T^{1/2}\log T\right). \tag{7}$$

In (7) $\gamma = -\Gamma'(1) = 0.5772157\ldots$ is Euler's constant. Jutila [12] obtained recently an asymptotic formula for the left-hand side of (7) when $0 < U \ll T^{1/2}$.

Bettin [1] evaluated asymptotically a related integral, namely

$$\int_0^T \zeta(1/2 + U + it)\zeta\left(\tfrac{1}{2} - V - it\right)\, dt$$

under the condition that

$$U = U(T) \in \mathbb{C},\ V = V(T) \in \mathbb{C},\ \Re e\, U \ll 1/\log T,\ \Re e\, V \ll 1/\log T.$$

Shimomura [16] dealt with the quartic moment

$$\int_0^T Z^2(t)Z^2(t+U)\, dt, \tag{8}$$

under certain conditions on the real parameter U, such that $|U| + 1/\log T \to 0$ as $T \to \infty$. When $U = 0$, Shimomura's expression for (8) reduces to

$$\int_0^T |\zeta\left(\tfrac{1}{2} + it\right)|^4\, dt = \int_0^T Z^4(t)\, dt = \frac{1}{2\pi^2} T\log^4 T + O(T\log^3 T). \tag{9}$$

The (weak) asymptotic formula (9) is a classical result of Ingham [4] of 1928. For the sharpest known asymptotic formula for the integral in (9), see the paper of Ivić–Motohashi [9].

2 Statement of Results

The first aim of this work is to investigate another integral with the shift of $Z(t)$, namely the cubic moment

$$\int_{T/2}^T Z^2(t)Z(t+U)\, dt \tag{10}$$

for a certain range of the "shift parameter" $U = U(T) > 0$. We shall prove the following

Theorem 1 *For $0 < U = U(T) \leqslant T^{1/2-\varepsilon}$ we have, uniformly in U,*

$$\int_{T/2}^T Z^2(t)Z(t+U)\, dt = O_\varepsilon(T^{3/4+\varepsilon})$$

$$+ 2\pi\sqrt{\frac{2}{3}} \sum_{T_1 \leqslant n \leqslant T_0} h(n,U) n^{-1/6+iU/3} \exp(-3\pi i n^{2/3} - \pi i/8)\Big\{1 + K(n,U)\Big\}. \tag{11}$$

Here ($d(n)$ is the number of divisors of n)

$$h(n,U) := n^{-iU}\sum_{\delta|n} d(\delta)\delta^{iU},\ T_0 := \frac{T^{3/2}}{\sqrt{8\pi^3}},\ T_1 := \frac{\left(\frac{T}{2}\right)^{3/2}}{\sqrt{8\pi^3}}, \tag{12}$$

$$K(n,U) := d_2U^2n^{-2/3} + \cdots + d_kU^kn^{-2k/3} + O_k(U^{k+1}n^{-2(k+1)/3}) \tag{13}$$

for any given integer $k \geqslant 2$, with effectively computable constants $d_2, d_3, \ldots$.

Remark 3 When $U \to 0+$, the expressions in (11)–(13) tend to (2), since the integral is real and $\lim_{U\to 0+} h(n,U) = d_3(n)$. Thus Theorem 1 is a generalization of (2) for a relatively wide range of U.

Remark 4 By methods similar to those used in proving Theorem 1 one can deal also with the "conjugate" integral

$$\int_{T/2}^{T} Z^2(t+U)Z(t)\,dt \qquad (0 < U \leqslant T^{1/2-\varepsilon}).$$

Remark 5 Analogously to (5) one may conjecture that, uniformly for $0 < U \leqslant T^{1/2-\varepsilon}$,

$$\int_0^T Z^2(t)Z(t+U)\,dt \ = \ O_\varepsilon(T^{3/4+\varepsilon}).$$

The second aim of this work is to demonstrate, although (5) and (6) seem at present intractable, that it is possible to show that the conjecture (6) holds in a certain mean value sense. To this end let

$$S(\alpha,N) := \sum_{N<n\leqslant N'\leqslant 2N} d_3(n)e^{\alpha i n^{2/3}} \qquad (\alpha \in \mathbb{R}), \tag{14}$$

so that the exponential sum in (6) is $S(3\pi,N)$. Then we have

Theorem 2 *Every finite interval $[A,B]$ $(0 < A < B)$ contains at least one point C such that*

$$S(C,N) \ \ll_\varepsilon \ N^{2/3}\log^{9/2}N. \tag{15}$$

3 Proof of the Theorems

We begin with the proof of Theorem 1. Suppose that $0 < U \leqslant T^{1/2-\varepsilon}$ and let

$$I := \int_{T/2}^{T} Z^2(t)Z(t+U)\,dt = \frac{1}{i}\int_{1/2+iT/2}^{1/2+iT} \zeta^2(s)\zeta(s+iU)\left(\chi^2(s)\chi(s+iU)\right)^{-1/2}ds. \tag{16}$$

The procedure of writing a real-valued integral like a complex integral is fairly standard in analytic number theory. For example, see the proof of Theorem 7.4 in Titchmarsh's monograph [17] on $\zeta(s)$ and Jutila's work [12]. It allows one flexibility by suitably deforming the contour of integration in the complex plane. Incidentally, this method of proof is different from the proof of (2) in [5], which is based on the use of approximate functional equations.

Let

$$h(n,U) := n^{-iU}\sum_{\delta|n} d(\delta)\delta^{iU}, \quad N_0 = N_0(T,U) := \sqrt{\frac{T^2(T+U)}{8\pi^3}}, \tag{17}$$

so that $h(n,U) \ll_\varepsilon n^\varepsilon$. We have

$$I = I_1 + I_2,$$

say, where

$$\begin{aligned} I_1 &:= \frac{1}{i}\int_{1/2+iT/2}^{1/2+iT} \sum_{n\leqslant N_0} h(n,U)n^{-s}(\chi^2(s)\chi(s+iU))^{-1/2}\,ds,\\ I_2 &:= \frac{1}{i}\int_{1/2+iT/2}^{1/2+iT} \left\{\zeta^2(s)\zeta(s+iU) - \sum_{n\leqslant N_0} h(n,U)n^{-s}\right\}(\chi^2(s)\chi(s+iU))^{-1/2}\,ds, \end{aligned} \tag{18}$$

and we shall keep in mind that for $\Re e\, s > 1$, since $d(n)$ is generated by $\zeta^2(s)$,

$$\sum_{n=1}^\infty h(n,U)n^{-s} = \sum_{k=1}^\infty\sum_{m=1}^\infty d(k)k^{iU}(km)^{-s}(km)^{-iU} = \zeta^2(s)\zeta(s+iU). \tag{19}$$

To be able to exploit (19) let henceforth $c = 1+\varepsilon$. Then, by Cauchy's theorem,

$$I_2 = \frac{1}{i}\left\{\int_{1/2+iT/2}^{c+iT/2} + \int_{c+iT}^{1/2+iT} + \int_{c+iT/2}^{c+iT}\right\} = \frac{1}{i}(J_1+J_2+J_3),$$

say. For $T/2 \leqslant t \leqslant T$, $\frac{1}{2} \leqslant \sigma \leqslant c$, $s = \sigma + it$, $0 < U \ll T^{1/2-\varepsilon}$ we have

$$\chi^2(s)\chi(s+iU) = e^{\frac{3\pi i}{4}}\left(\frac{2\pi}{t}\right)^{2\sigma+2it-1}\left(\frac{2\pi}{t+U}\right)^{\sigma+it+iU-\frac{1}{2}} e^{3it+iU}\left\{1+O\left(\frac{1}{t}\right)\right\}, \tag{20}$$

so that uniformly in U

$$\chi^2(s)\chi(s+iU) \asymp T^{3/2-3\sigma} \qquad (t \asymp T).$$

This follows from the asymptotic formula (1), and actually the term $O(1/t)$ can be replaced by an asymptotic expansion. We also have the convexity bound (see, e.g., Chap. 1 of [5])

$$\zeta(s) \ll_\varepsilon |t|^{(1-\sigma)/2+\varepsilon} \qquad (\tfrac{1}{2} \leqslant \sigma \leqslant c). \tag{21}$$

Stronger bounds than (21) are available, but this bound is sufficient for our present purposes.

By using (20), (21) and the trivial bound

$$\sum_{n\leqslant N_0} h(n,U)n^{-s} \ll_\varepsilon N_0^{1-\sigma+\varepsilon} \ll_\varepsilon T^{(3-3\sigma)/2+\varepsilon} \quad \left(\tfrac{1}{2} \leqslant \sigma \leqslant c\right),$$

it follows readily that

$$J_1 + J_2 \ll_\varepsilon \int_{1/2}^{c} T^{(3-3\sigma)/2+\varepsilon} T^{(6\sigma-3)/4}\, d\sigma \ll_\varepsilon T^{3/4+\varepsilon}. \tag{22}$$

In J_3 we have $s = c + it$, $T/2 \leqslant t \leqslant T$. Hence

$$\begin{aligned} J_3 =& i\sum_{n>N_0} h(n,U)n^{-c} \int_{T/2}^{T} n^{-it} e^{-3\pi i/8} \left(\frac{t}{2\pi}\right)^{c+it-1/2} \\ &\times \left(\frac{t+U}{2\pi}\right)^{(2c+2it+2iU-1)/4} e^{-(3it+iU)/2}\, dt \\ &+ O\left(\sum_{n>N_0} |h(n,U)| n^{-c} T^{(6c-3)/4}\right). \end{aligned}$$

The expression in the O-term is clearly $\ll_\varepsilon T^{3/4+\varepsilon}$, and it remains to estimate

$$\sum_{n>N_0} h(n,U)n^{-c} e^{-iU/2} \int_{T/2}^{T} K(t) e^{if_n(t)}\, dt, \tag{23}$$

with

$$K(t) := \left(\frac{t}{2\pi}\right)^{c-1/2} \left(\frac{t+U}{2\pi}\right)^{(2c-1)/4} \ll_\varepsilon T^{3/4+\varepsilon}$$

for $c = 1 + \varepsilon$. In (23) we have

$$f_n(t) := t\log\frac{t}{2\pi} + \tfrac{1}{2}(t+U)\log\frac{t+U}{2\pi} - \tfrac{3}{2}t - t\log n,$$

$$f_n'(t) = \log\frac{t}{2\pi} + \tfrac{1}{2}\log\frac{t+U}{2\pi} - \log n,$$
$$f_n''(t) = \frac{1}{t} + \frac{1}{2(t+U)} = \frac{3t+2U}{2t(t+U)} \asymp \frac{1}{T}. \tag{24}$$

In the sum in (23) set $n = [N_0] + \nu$, $\nu \in \mathbb{N}$. By the second derivative test (Lemma 2.2 of [5]) the contribution of the term $\nu = 1$ is $\ll T^{\varepsilon-1/4}$. The terms $\nu \geqslant 2$ are estimated by the use of the first derivative test (Lemma 2.1 of [5]). For $\nu \gg N_0$ it is found that $|f_n'(t)| \gg 1$, while for $\nu \ll N_0$ one has, for a suitable $c > 0$,

$$|f_n'(t)| \;=\; \log\frac{[N_0]+\nu}{\frac{t}{2\pi}\sqrt{\frac{t+U}{2\pi}}} \;\geqslant\; \log\bigl(1 + c\nu T^{-3/2}\bigr) \gg \nu T^{-3/2}.$$

Thus the contribution of the terms with $\nu \geqslant 2$ will be

$$\begin{aligned}
&\ll_\varepsilon T^{3/4+\varepsilon}\left(\sum_{2\leqslant\nu\ll N_0} + \sum_{\nu\gg N_0}\right)|h([N_0]+\nu, U)|n^{-c}\max_{T/2\leqslant t\leqslant T}|f_n'(t)|^{-1}\\
&\ll_\varepsilon T^{3/4+\varepsilon}\left(\sum_{2\leqslant\nu\ll N_0}(N_0+\nu)^{-1-\varepsilon}T^{3/2}\nu^{-1} + \sum_{\nu\gg N_0}\nu^{\varepsilon/2-1-\varepsilon}\right)\\
&\ll_\varepsilon T^{3/4+\varepsilon},
\end{aligned}$$

since $N_0 \asymp T^{3/2}$.

It remains to deal with I_1. We have

$$\begin{aligned}
I_1 &= e^{-3\pi i/8}\int_{T/2}^{T}\sum_{n\leqslant N_0} h(n,U)n^{-1/2}e^{-iU/2}e^{if_n(t)}\,dt + O_\varepsilon(T^{3/4+\varepsilon})\\
&= e^{-iU/2}e^{-3\pi i/8}\sum_{n\leqslant N_0} h(n,U)n^{-1/2}\,\mathscr{J}_n(T) + O_\varepsilon(T^{3/4+\varepsilon}),
\end{aligned} \tag{25}$$

say, where

$$\mathscr{J}_n(T) \;:=\; \int_{T/2}^{T} e^{if_n(t)}\,dt. \tag{26}$$

The exponential integral in (26) has a saddle point t_n, the root of the equation $f_n'(t) = 0$, which occurs for

$$t_n^2(t_n + U) = 8\pi^3 n^2. \tag{27}$$

It is immediately seen that $t_n \asymp n^{2/3} \asymp T$, and that Eq. (27) is cubic in t_n for a given n and U, and thus it is not easily solvable. However, its solution can be found asymptotically by an iterative process. First from (27) we have

$$t_n^3 = 8\pi^3 n^2 + O(UT^2),$$

which yields

$$t_n = 2\pi n^{2/3} + O(U). \tag{28}$$

As we require $T/2 \leqslant t_n \leqslant T$, from (27) we infer that $N_1 \leqslant n \leqslant N_0$, where N_0 is defined by (17) (this is the reason for the choice of N_0) and

$$N_1 := \sqrt{\frac{(T/2)^2(T/2+U)}{8\pi^3}}.$$

Setting $n = [N_1] - \mu, 1 \leqslant \mu < N_1$ and proceeding similarly as in the estimation of the sum with $\nu \geqslant 2$ after (24), we obtain that the contribution of $n \leqslant N_1$ to I_1 in (25) will be $O_\varepsilon(T^{3/4+\varepsilon})$. Since $\mathscr{J}_n(T) \ll T^{1/2}$ [see (31) and (32)] it transpires that the condition $N_1 \leqslant n \leqslant N_0$ may be replaced by $T_1 \leqslant n \leqslant T_0$, where T_0, T_1 are defined by (12). In this process the total error term will be $O_\varepsilon(T^{3/4+\varepsilon})$. Therefore, for the range $0 < U \leqslant T^{1/2-\varepsilon}$, we have uniformly

$$I_1 = e^{-iU/2}e^{-3\pi i/8} \sum_{T_1 \leqslant n \leqslant T_0} h(n,U) n^{-1/2} \mathscr{J}_n(T) + O_\varepsilon(T^{3/4+\varepsilon}), \tag{29}$$

and the range of summation in (29) is easier to deal with than the range of summation $N_1 < n \leqslant N_0$. At this point we shall evaluate $\mathscr{J}_n(T)$ by one of the theorems related to exponential integrals with a saddle point, e.g., by the one on p. 71 of the monograph by Karatsuba–Voronin [13]. This says that, if $f(x) \in C^{(4)}[a,b]$,

$$\begin{aligned}\int_a^b e^{2\pi i f(x)}\,dx =& e^{\pi i/4}\frac{e^{2\pi i f(c)}}{\sqrt{f''(c)}} + O(AV^{-1}) \\ &+ O\Big(\min(|f'(a)|^{-1}, \sqrt{a}\,)\Big) + O\Big(\min(|f'(b)|^{-1}, \sqrt{b}\,)\Big),\end{aligned} \tag{30}$$

where

$$\begin{aligned}&0 < b-a \leqslant V,\ f'(c) = 0,\ a \leqslant c \leqslant b, \\ &f''(x) \asymp A^{-1},\ f^{(3)}(x) \ll (AV)^{-1},\ f^{(4)}(x) \ll A^{-1}V^{-2}.\end{aligned}$$

We shall apply (30) with

$$f(x) = \frac{f_n(x)}{2\pi},\ c = t_n,\ a = \tfrac{1}{2}T,\ b = T,\ V = \tfrac{1}{2}T, f''(x) \asymp \frac{1}{T},$$

so that $A = T, f^{(3)}(x) \ll T^{-2}, f^{(4)}(x) \ll T^{-3}$, which is needed. The term $O(AV^{-1})$ in (30) will make a contribution which is $O_\varepsilon(T^{3/4+\varepsilon})$, and the same assertion will hold for the other two error terms in (30). This will lead to

$$I_1 = e^{-iU/2}e^{-3\pi i/8}\sum_{T_1\leq n\leq T_0} h(n,U)n^{-1/2}e^{\pi i/4}\frac{e^{if_n(t_n)}}{\sqrt{\frac{f_n''(t_n)}{2\pi}}} + O_\varepsilon(T^{3/4+\varepsilon}). \tag{31}$$

Since, by (24) and (27),

$$\begin{aligned} e^{if_n(t)} &= \exp\Big\{i\Big(t_n\log\frac{t_n}{2\pi} + \tfrac12(t_n+U)\log\frac{(t_n+U)}{2\pi} - \tfrac32 t_n - t_n\log n\Big)\Big\}\\ &= \exp\Big\{\tfrac12 it_n\log\frac{t_n^2(t_n+U)}{8\pi^3} + \tfrac12 Ui\log\frac{(t_n+U)}{2\pi} - \tfrac32 it_n - it_n\log n\Big\}\\ &= \exp\Big\{\tfrac12 iU\log\frac{(t_n+U)}{2\pi} - \tfrac32 it_n\Big\}\\ &= e^{-3it_n/2}\Big(\frac{t_n+U}{2\pi}\Big)^{iU/2}, \end{aligned}$$

it follows that

$$\begin{aligned} I_1 =& O_\varepsilon(T^{3/4+\varepsilon}) + \sqrt{2\pi}e^{-iU/2}e^{-\pi i/8}\\ &\times\sum_{T_1\leq n\leq T_0} h(n,U)n^{-1/2}\sqrt{\frac{2t_n(t_n+U)}{3t_n+2U}}e^{-3it_n/2}\Big(\frac{t_n+U}{2\pi}\Big)^{iU/2}. \end{aligned} \tag{32}$$

In view of (27), (28) and $n \asymp T^{3/2}$ we have

$$\begin{aligned} n^{-1/2}\sqrt{\frac{2t_n(t_n+U)}{3t_n+2U}} &= n^{1/2}\sqrt{\frac{2(2\pi)^3}{t_n(3t_n+2U)}}\\ &= \sqrt{2(2\pi)^3}n^{1/2}t_n^{-1}\Big(3^{-1/2}+O(U/T)\Big)\\ &= \sqrt{\frac23(2\pi)^3}n^{1/2}\Big((8\pi^3n^2)^{1/3}+O(U)\Big)^{-1} + O(UT^{-5/4})\\ &= 2\sqrt{\frac{\pi}{3}}n^{-1/6} + O(UT^{-5/4}). \end{aligned}$$

It remains to evaluate

$$\mathscr{A} := \mathscr{A}(n,U) = e^{-iU/2}e^{-3it_n/2}\Big(\frac{t_n+U}{2\pi}\Big)^{iU/2}. \tag{33}$$

To achieve this we need a more precise expression for t_n than (28). Putting (28) in (27) we first have

$$\begin{aligned} t_n^3 &= 8\pi^3 n^2 - Ut_n^2 = 8\pi^3 n^2 - 4\pi^2 U n^{4/3} + O(n^{2/3}U^2), \\ t_n &= (8\pi^3 n^2 - 4\pi^2 U n^{4/3})^{1/3}\left(1 + O(U^2 n^{-4/3})\right) \\ &= 2\pi n^{2/3} - \tfrac{1}{3}U + O(U^2 n^{-2/3}). \end{aligned}$$

This yields the second approximation

$$t_n = 2\pi n^{2/3} - \tfrac{1}{3}U + O(U^2 n^{-2/3}). \tag{34}$$

Inserting (34) in (27) we shall obtain next

$$t_n = 2\pi n^{2/3} - \tfrac{1}{3}U - \tfrac{U^2 n^{-2/3}}{18\pi} + O(U^3 n^{-4/3}). \tag{35}$$

In fact, (35) is just the beginning of an asymptotic expansion for t_n in which each term is by a factor of $Un^{-2/3}$ of a lower order of magnitude than the preceding one. This can be established by mathematical induction. Then we see that (c's and d's are effectively computable constants)

$$\begin{aligned} \log \mathscr{A} &= i\Big\{ -\tfrac{1}{2}U - \frac{3}{2}(2\pi)n^{2/3} + \tfrac{1}{2}U + \frac{U^2 n^{-2/3}}{12\pi} + c_3 U^3 n^{-4/3} + O(U^4 n^{-6/3}) \\ &\quad + \tfrac{1}{2}U \log\Big(n^{2/3} + \frac{U}{3\pi} - \frac{U^2 n^{-2/3}}{36\pi^2} + O(U^3 n^{-4/3})\Big)\Big\} \\ &= -3\pi i n^{2/3} + \tfrac{1}{3}iU \log n + ic_2 U^2 n^{-2/3} + ic_3 U^3 n^{-4/3} + O(U^4 n^{-6/3}), \end{aligned}$$

and therefore

$$\mathscr{A} = n^{iU/3} e^{-3\pi i n^{2/3}} \left\{1 + id_2 U^2 n^{-2/3} + id_3 U^3 n^{-4/3} + O(U^4 n^{-6/3})\right\}, \tag{36}$$

and we remark that the last term in curly brackets admits an asymptotic expansion. Since $U \ll T^{1/2-\varepsilon}$ by assumption, $U^2 n^{-2/3} \ll T^{-2\varepsilon}$ and thus each term in the expansion will be by a factor of at least $T^{-2\varepsilon}$ smaller than the preceding one. One sees now why the interval of integration in (10) and (11) is $[T/2, T]$ and not the more natural $[0, T]$, because the terms $d_j U^j n^{-2j/3}$ become large if n is small, and $K(n, U)$ is then also large.

Inserting (36) (as a full asymptotic expansion) and (33) in (32), we obtain the assertion of Theorem 1, having in mind that we have shown that

$$|I_2| + |I_3| \ll_\varepsilon T^{3/4+\varepsilon}.$$

To prove Theorem 2 note that, with $S(\alpha, N)$ defined by (14), we have

$$\int_A^B |S(\alpha,N)|^2\,d\alpha = \int_A^B \left(\sum_{n\asymp N} d_3^2(n) + \sum_{m\ne n\asymp N} d_3(m)d_3(n)e^{i\alpha(m^{2/3}-n^{2/3})}\right) d\alpha$$

$$= (B-A)\sum_{n\asymp N} d_3^2(n) + O\left(\sum_{m\ne n\asymp N}\frac{d_3(m)d_3(n)}{|m^{2/3}-n^{2/3}|}\right). \tag{37}$$

Note that the function $d_3(n)$ is multiplicative and $d_3(p) = 3$. Thus for $\Re e\, s > 1$ and p a generic prime we have

$$\sum_{n=1}^\infty d_3^2(n)n^{-s} = \prod_p \left(1 + d_3^2(p)p^{-s} + d_3^2(p^2)p^{-2s} + \cdots\right)$$

$$= \zeta^9(s)\prod_p (1-p^{-s})^9 \prod_p \left(1 + 9p^{-s} + d_3^2(p^2)p^{-2s} + \cdots\right)$$

$$= \zeta^9(s)F(s), \tag{38}$$

where $F(s)$ is a Dirichlet series that is absolutely convergent for $\Re e\, s > 1/2$. Thus the series in (38) is dominated by $\zeta^9(s)$ and it follows, by a simple convolution argument, that

$$\sum_{n\leqslant x} d_3^2(n) \;=\; C_1 x\log^8 x \;+\; O(x\log^7 x) \tag{39}$$

for a constant $C_1 > 0$.

Next, by using the trivial inequality

$$|ab| \;\leqslant\; \tfrac{1}{2}(a^2+b^2)$$

and (39), it is seen that the double sum in (37) in the O-term is, by symmetry,

$$\ll N^{1/3}\sum_{m\ne n\asymp N}\frac{d_3^2(m)+d_3^2(n)}{|m-n|}$$

$$= 2N^{1/3}\sum_{m\asymp N} d_3^2(m)\sum_{n\ne m, n\asymp N}\frac{1}{|n-m|}$$

$$\ll N^{4/3}\log^9 N,$$

since, for a fixed m such that $m \asymp N$,

$$\sum_{n\ne m, n\asymp N}\frac{1}{|n-m|} \;\ll\; \log N.$$

From this estimate and (39) it follows that

$$\int_A^B |S(\alpha, N)|^2 \, d\alpha \ll N^{4/3} \log^9 N, \tag{40}$$

which implies the assertion of Theorem 2. A further discussion on the cubic moment of $Z(t)$ is to be found in Chap. 11 of the author's monograph [8].

Remark 6 An interesting problem is to determine the true order of the integral in (40).

Remark 7 If the divisor function $d_3(n)$ is not present in the definition (14), namely if one considers the sum

$$T(\alpha, N) := \sum_{N < n \leq N' \leq 2N} e^{\alpha i n^{2/3}} \qquad (\alpha \neq 0),$$

then the problem is much easier. By using Lemma 1.2 and Lemma 2.1 of [5] it easily follows that

$$T(\alpha, N) \ll |\alpha|^{-1} N^{1/3}.$$

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Some Analogues of Pair Correlation of Zeta Zeros

Yunus Karabulut and Cem Yalçın Yıldırım

Abstract By modifying Montgomery's calculation of the pair correlation of zeta zeros, we derive analogous results. In this article we work out the correlation of zeta zeros with the relative maxima of the zeta-function on the critical line, the pair correlation of these maxima and the correlation of zeros of two Dirichlet L-functions. In each case the relevant Riemann Hypothesis is assumed for obtaining the results. Several auxiliary results necessary for the calculations may be useful in problems about the zeta-function.

1 Introduction

The Riemann zeta-function $\zeta(s)$, for the complex variable $s = \sigma + it$ with real numbers σ and t, has all of its so-called non-trivial zeros in the strip $\sigma \in (0, 1)$, and the number of these zeros with ordinates in $(0, T]$ is

$$N_\zeta(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{1}$$

The Riemann Hypothesis (RH) is the statement that all of these zeros are on the critical line $\sigma = \frac{1}{2}$. In [17] Montgomery, starting from an explicit formula (given in (18) below) in Landau's Handbuch [16] and assuming RH, elicited and defined the expression

$$F_{\zeta,\zeta}(x, T) := \sum_{0<\gamma,\tilde{\gamma}\le T} \frac{4}{4 + (\gamma - \tilde{\gamma})^2}\, x^{i(\gamma-\tilde{\gamma})}, \quad (x > 0), \tag{2}$$

calculated its values, and broke new ground by the results, conjectures and observations concerning the distribution of zeta zeros on the critical line and relations with the distribution of primes which were deduced therefrom. Here γ and $\tilde{\gamma}$ each denote

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H. Montgomery et al. (eds.), *Exploring the Riemann Zeta Function*,
DOI 10.1007/978-3-319-59969-4_7

the ordinate of a non-trivial zero of $\zeta(s)$, and $\frac{4}{4+u^2}$ is a suitable weight function. From the definition (2) it is clear that $F_{\zeta,\zeta}(x,T) = F_{\zeta,\zeta}\left(\frac{1}{x},T\right)$ (so that one can consider only $x \geq 1$), and that

$$F_{\zeta,\zeta}(x,T) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left|\sum_{0<\gamma\leq T}\frac{x^{i\gamma}}{1+(t-\gamma)^2}\right|^2 dt \geq 0. \tag{3}$$

Montgomery's calculation gave

$$F_{\zeta,\zeta}(x,T) = \frac{T\log^2 T}{2\pi x^2}\left(1+O\left(\frac{1}{\log T}\right)\right)+\frac{T}{2\pi}\log x + o(T\log T)+O(x\log x), \tag{4}$$

as $T\to\infty$. In [17] Montgomery expressed his results in a different form which is related to (2) by setting

$$x = T^{\alpha},\ \ T\geq 2,\ \ \alpha\in\mathbb{R} \tag{5}$$

and

$$F(\alpha) = F(\alpha,T) := \left(\frac{T}{2\pi}\log T\right)^{-1}\sum_{0<\gamma,\tilde{\gamma}\leq T} T^{i\alpha(\gamma-\tilde{\gamma})}w(\gamma-\tilde{\gamma}), \tag{6}$$

so that the result read

$$F(\alpha) = (1+o(1))T^{-2\alpha}\log T+\alpha+o(1), \tag{7}$$

as $T\to\infty$, uniformly for $0\leq\alpha\leq 1$, along with $F(-\alpha) = F(\alpha)$. (With further effort involving the use of a sieve upper bound for prime pairs, Goldston [8] improved the error term $O(x\log x)$ in (4) to $O(x)$, as had been pointed out by Montgomery, leading to the uniformity up to $\alpha = 1$; see also [12]).

By convolving $F(\alpha)$ with appropriate kernels, Montgomery deduced that asymptotically at least $\frac{2}{3}$ of zeta zeros are simple, and the existence of infinitely many gaps between consecutive zeta zeros of size at most 0.68 times the average gap. This has since been reduced to 0.51... by several researchers; achieving an upper bound below $\frac{1}{2}$ will have important consequences in number theory (see [15]). By drawing upon conjectural estimates for twin primes, and also upon the q-analogue of an expression which comes up in his calculation, Montgomery conjectured that

$$F(\alpha) = 1+o(1) \tag{8}$$

uniformly for $\alpha\geq 1$ in bounded intervals. Trivially $F(\alpha)\leq F(0)$ for any α, and by Dirichlet's theorem on Diophantine approximation, there are large $\alpha = \alpha(T)$ such that $F(\alpha)$ becomes arbitrarily close to $F(0)\sim\log T$ (which follows unconditionally

from (1)), so the restriction to bounded intervals is essential for (8). We note that such large values of $F(\alpha)$ can occur over intervals of length at most $O\big(\frac{1}{\log T}\big)$. This follows from the upper bound in Goldston's (see [9]) estimates that for any $\epsilon > 0$ and sufficiently large T

$$\int_{c-1}^{c+1} F(\alpha)\, d\alpha \geq \frac{2}{3} - \epsilon, \quad \int_{c}^{c+1} F(\alpha)\, d\alpha \leq \frac{8}{3} + \epsilon, \tag{9}$$

uniformly for any real number c (which may be any function of T). From the conjecture (8), Montgomery deduced the so-called pair correlation conjecture

$$\sum_{\substack{0<\gamma,\tilde{\gamma}\leq T \\ \frac{2\pi\alpha}{\log T} \leq \gamma-\tilde{\gamma} \leq \frac{2\pi\beta}{\log T}}} 1 \quad \sim \quad \left(\int_{\alpha}^{\beta} 1 - \left(\frac{\sin \pi u}{\pi u} \right)^2 du + \delta(\alpha,\beta) \right) \frac{T}{2\pi} \log T, \tag{10}$$

as $T \to \infty$. Here $\delta(\alpha,\beta) = 1$ if $0 \in [\alpha,\beta]$, $\delta(\alpha,\beta) = 0$ otherwise. As for the converse direction, Goldston [10] showed that (10) implies $\int_{\alpha}^{\alpha+\delta} F(\beta,T)\, d\beta \sim \delta$ for any fixed numbers $\alpha \geq 1$ and $\delta > 0$, i.e. that (8) holds on average. From either (8) or (10), it follows that almost all zeta zeros are simple. F.J. Dyson remarked to Montgomery that the integrand in (10) occurs as the pair correlation function of the eigenvalues of random complex Hermitian matrices in the Gaussian Unitary Ensemble, and Montgomery interpreted this being in accordance with the view (which legend dates back to Hilbert and Pólya, see [20]) that there is a yet undiscovered linear operator whose eigenvalues characterize the zeros of the Riemann zeta-function. Montgomery's work [17] has been very influential for the developments in the theory of the Riemann zeta function (and other zeta functions as well) and formed the basis of the connections with random matrix theory. For a comprehensive survey on this topic and relations with the distribution of primes we refer the reader to Goldston's article [11].

Brief Statement of Results In this article we modify Montgomery's argument in such a way that we can obtain some analogues of the pair correlation of zeta zeros. We begin with a brief sketch of Montgomery's method. Then in Sect. 3 we present an alternative method for correlating zeta zeros to any sequence of real numbers. The first application, in Sect. 4, is to Montgomery's original problem. (The contents of Sects. 3 and 4 were already announced in [24]). Upon some preliminaries concerning $\chi(s)$ (defined in (12)) and the relative maxima of $\zeta(s)$ on the critical line in Sect. 5 and concerning the iterated convolutions of the von Mangoldt function in Sect. 6, the method of Sect. 3 is applied in Sect. 7 to the correlation of zeta zeros with the maxima of zeta on the critical line. Corresponding to $F(\alpha) \sim \alpha$ for $\epsilon \leq \alpha \leq 1-\epsilon$ from (7), we find in Theorem 1 in Sect. 7 that $F_{\zeta,Z_1}(\alpha) \sim \alpha - 2\alpha^2$, $(\epsilon \leq \alpha \leq 1-\epsilon)$. Here $Z_1(s)$, defined in Sect. 5, is a function which vanishes at the relative maxima of $\zeta(s)$ on the critical line. For this purpose

a Landau–Gonek type formula involving the zeros of $Z_1(s)$ is developed in Sect. 7. In Theorem 2 in Sect. 8 we obtain the values of the pair correlation of the maxima $F_{Z_1,Z_1}(\alpha)$ for $0 \leq \alpha \leq 1-\epsilon$, to be followed by values of $F_{\zeta\cdot Z_1,\zeta\cdot Z_1}(\alpha)$, and then deduce some results on small gaps. In Sect. 9 we adapt our alternative argument to obtain in Theorem 3 the correlation between the zeros of two Dirichlet L-functions. Finally in Sect. 10 we mention some possible ways to continue this research.

Some Basic Formulae Before embarking on the calculations we recall that the Riemann zeta-function satisfies the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{11}$$

where

$$\chi(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right) = \pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}. \tag{12}$$

Logarithmic differentiation of the functional equation gives

$$\frac{\zeta'}{\zeta}(s) = \frac{\chi'}{\chi}(s) - \frac{\zeta'}{\zeta}(1-s). \tag{13}$$

From (12) and some basic properties of the Γ-function, we see that

$$\frac{\chi'}{\chi}(s) = -\frac{\Gamma'}{\Gamma}(s) + \frac{\pi}{2}\tan\frac{s\pi}{2} + \log 2\pi = -\frac{\Gamma'}{\Gamma}(1-s) + \frac{\pi}{2}\cot\frac{s\pi}{2} + \log 2\pi. \tag{14}$$

Since

$$\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} + O\left(\frac{1}{|s|^2}\right), \quad (|s| \geq \delta > 0,\ |\arg s| \leq \pi - \delta), \tag{15}$$

where $\delta > 0$ is arbitrarily small but fixed, we have

$$\frac{\chi'}{\chi}(s) = -\log\frac{|s|}{2\pi} + O\left(\frac{1}{|s|}\right), \quad (|t| \geq 1, \text{ say; } \sigma_1 \leq \sigma \leq \sigma_2), \tag{16}$$

where $\sigma_1 < \sigma_2$ are any fixed real numbers. Under the same conditions one also has

$$\left(\frac{d}{ds}\right)^n \frac{\chi'}{\chi}(s) \ll |s|^{-n}, \quad (n \geq 1). \tag{17}$$

Some further facts concerning $\zeta(s)$ and $\chi(s)$ will be given in Sects. 5 and 6.

Notation Throughout this paper, T will denote a large number tending to ∞, ϵ will be a fixed positive number which can be taken arbitrarily small and which may not have the same value each time it appears, and letters with indices such as C_j will denote sufficiently large positive constants of fixed value. We write $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$. We will employ the Iverson notation that for a statement S, $[S] = 1$ if S is true, and $[S] = 0$ if S is false.

2 A Sketch of Montgomery's Derivation

The explicit formula from Landau's book [16] is that, if $x > 1$ and $x \neq p^k$ (p prime, and k a positive integer), then

$$\sum_{n\leq x} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'}{\zeta}(s) + \frac{x^{1-s}}{1-s} - \sum_{\rho} \frac{x^{\rho-s}}{\rho-s} + \sum_{r=1}^{\infty} \frac{x^{-2r-s}}{2r+s} \tag{18}$$

provided $s \neq 1$, $s \neq \rho$, $s \neq -2r$. Here ρ runs through the non-trivial zeros of $\zeta(s)$, and the series over ρ is convergent subject to the interpretation $\lim\limits_{U\to\infty} \sum\limits_{|\Im\,\rho|\leq U} \dfrac{x^{\rho-s}}{\rho-s}$.

Upon assuming RH, Montgomery deduced that

$$\begin{aligned}(2\sigma-1)\sum_{\gamma} \frac{x^{i\gamma}}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2} &= -x^{-\frac{1}{2}}\left(\sum_{n\leq x}\Lambda(n)\left(\frac{x}{n}\right)^{1-\sigma+it} \sum_{n>x}\Lambda(n)\left(\frac{x}{n}\right)^{\sigma+it}\right)\\ &\quad -\frac{\zeta'}{\zeta}(1-\sigma+it)x^{\frac{1}{2}-\sigma+it} + \frac{(2\sigma-1)x^{\frac{1}{2}}}{(\sigma-1+it)(\sigma-it)}\\ &\quad -x^{-\frac{1}{2}}\sum_{r=1}^{\infty}\frac{(2\sigma-1)x^{-2r}}{(\sigma-1-it-2r)(\sigma+it+2r)},\end{aligned} \tag{19}$$

valid for $\sigma > 1$, and all $x \geq 1$. In this formula Montgomery used (13)–(16) to replace $\frac{\zeta'}{\zeta}(1-\sigma+it)$ by $-\frac{\zeta'}{\zeta}(\sigma-it) - \log(|t|+2) + O(1) = -\log(|t|+2) + O_\sigma(1)$ (for s in a fixed strip in $\sigma > 1$), and he put upper-bound estimates for the last two terms of (19). Montgomery took $\sigma = \frac{3}{2}$, squared the modulus of both sides, and then integrated both sides over t from 0 to T. From the expression thus arising from the left-hand side of (19), he discarded those $\gamma \notin [0, T]$ within an error of $O(\log^3 T)$, and in order to evaluate the integral he extended the range of integration to $\int_{-\infty}^{\infty}$ within an even smaller error. In this way the left-hand side of (19) led to the expression (2). To carry out the integration of the square of the series involving $\Lambda(n)$ coming from the right-hand side of (19), Montgomery resorted to Parseval's formula for Dirichlet series from [18]:

$$\text{If } \sum_{n=1}^{\infty} n|a_n|^2 \text{ converges, then } \int_0^T \left|\sum_n a_n n^{-it}\right|^2 dt = \sum_n |a_n|^2 (T + O(n)),$$

The result of this calculation was (4) (or (7)).

3 A Modified Approach

With $\sigma = \frac{5}{2}$, (19) reads

$$\sum_{\gamma} \frac{4x^{i\gamma}}{4+(t-\gamma)^2} = -x^{-\frac{1}{2}} \left(\sum_{n \le x} \Lambda(n) \left(\frac{x}{n} \right)^{-\frac{3}{2}+it} + \sum_{n>x} \Lambda(n) \left(\frac{x}{n} \right)^{\frac{5}{2}+it} \right)$$
$$-\frac{\zeta'}{\zeta}\left(-\frac{3}{2} + it \right) x^{-2+it} + \frac{4x^{\frac{1}{2}}}{\left(\frac{3}{2}+it\right)\left(\frac{5}{2}-it\right)}$$
$$-4x^{-\frac{1}{2}} \sum_{r=1}^{\infty} \frac{x^{-2r}}{\left(\frac{3}{2}-2r-it\right)\left(\frac{5}{2}+2r+it\right)}.$$

Using

$$\frac{\zeta'}{\zeta}(1-\sigma+it) = -\frac{\zeta'}{\zeta}(\sigma-it) + \log\pi + \frac{1}{(\sigma-it)(1-\sigma+it)}$$
$$-\frac{1}{2}\left(\frac{\Gamma'}{\Gamma}\left(\frac{3-\sigma+it}{2} \right) + \frac{\Gamma'}{\Gamma}\left(1 + \frac{\sigma-it}{2} \right) \right) \tag{20}$$

which follows from the formulas in Chap. 12 of [2], with $\sigma = \frac{5}{2}$, we have

$$\sum_{\gamma} \frac{4x^{i(\gamma-t)}}{4+(\gamma-t)^2} = -x^{-2} \sum_{n \le x} \Lambda(n) n^{\frac{3}{2}-it} - x^2 \sum_{n>x} \Lambda(n) n^{-\frac{5}{2}-it}$$
$$+x^{-2}\left(\frac{\zeta'}{\zeta}\left(\frac{5}{2} - it \right) - \log\pi \right)$$
$$+\frac{x^{-2}}{2}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{it}{2} \right) + \frac{\Gamma'}{\Gamma}\left(\frac{9}{4} - \frac{it}{2} \right) \right)$$
$$+\frac{x^{-2}}{\left(\frac{5}{2}-it\right)\left(\frac{3}{2}-it\right)} + \frac{4x^{\frac{1}{2}-it}}{\left(\frac{3}{2}+it\right)\left(\frac{5}{2}-it\right)}$$
$$-4x^{-\frac{1}{2}-it} \sum_{r=1}^{\infty} \frac{x^{-2r}}{\left(\frac{3}{2}-2r-it\right)\left(\frac{5}{2}+2r+it\right)}. \tag{21}$$

Using (15), and doing elementary estimations, we can simplify (21) into

$$\sum_{\gamma}\frac{4x^{i(\gamma-t)}}{4+(\gamma-t)^2}=-x^{-2}\sum_{n\le x}\Lambda(n)n^{\frac{3}{2}-it}-x^2\sum_{n>x}\Lambda(n)n^{-\frac{5}{2}-it}$$
$$+x^{-2}\big(\log(|t|+3)+O(1)\big)$$
$$+O\left(\frac{x^{\frac{1}{2}}}{(|t|+1)^2}\right)+O\left(\frac{x^{-\frac{5}{2}}}{|t|+1}\right),\quad (x\ge 1). \tag{22}$$

When t is let to run through a set we sum both sides of (22) over this set of values of t. This will be feasible if one can calculate the sums over t, in particular $\sum_t p^{-iat}$ where p is a prime and a is a natural number.

4 Pair Correlation of Zeta Zeros

We apply our method first to the quantity (2) considered by Montgomery. So, letting t run through those ordinates $\tilde{\gamma}$ of the zeros of the Riemann zeta-function which are in the interval $(0,T]$, from (22) we have

$$\sum_{\gamma}\sum_{0<\tilde{\gamma}\le T}\frac{4x^{i(\gamma-\tilde{\gamma})}}{4+(\gamma-\tilde{\gamma})^2}=-x^{-2}\sum_{n\le x}\Lambda(n)n^{\frac{3}{2}}\sum_{0<\tilde{\gamma}\le T}n^{-i\tilde{\gamma}}-x^2\sum_{n>x}\frac{\Lambda(n)}{n^{\frac{5}{2}}}\sum_{0<\tilde{\gamma}\le T}n^{-i\tilde{\gamma}}$$
$$+x^{-2}\sum_{0<\tilde{\gamma}\le T}\big(\log\tilde{\gamma}+O(1)\big)+O\left(x^{\frac{1}{2}}\sum_{0<\tilde{\gamma}\le T}\frac{1}{\tilde{\gamma}^2}\right)$$
$$+O\left(x^{-\frac{5}{2}}\sum_{0<\tilde{\gamma}\le T}\frac{1}{\tilde{\gamma}}\right) \tag{23}$$

for $x\ge 1$. From the count of zeta zeros with ordinates in $(0,T]$ given in (1), we easily estimate the last terms of (23) and re-state (23) as

$$\sum_{\gamma}\sum_{0<\tilde{\gamma}\le T}\frac{4x^{i(\gamma-\tilde{\gamma})}}{4+(\gamma-\tilde{\gamma})^2}=-x^{-2}\sum_{n\le x}\Lambda(n)n^{\frac{3}{2}}\sum_{0<\tilde{\gamma}\le T}n^{-i\tilde{\gamma}}-x^2\sum_{n>x}\frac{\Lambda(n)}{n^{\frac{5}{2}}}\sum_{0<\tilde{\gamma}\le T}n^{-i\tilde{\gamma}}$$
$$+\frac{T\log^2 T}{2\pi x^2}\left(1+O\left(\frac{1}{\log T}\right)\right)+O(x^{\frac{1}{2}}). \tag{24}$$

In the sum on the left-hand side we can exclude those $\gamma\notin[0,T]$ within an error of $O(\log^3 T)$, by a standard calculation based upon (1).

We now have recourse to Gonek's [13] uniform version of Landau's formula

$$\sum_{0<\gamma\le T} y^{\rho} = -\frac{T}{2\pi}\Lambda(y) + O(y\log 2yT\log\log 3y) \tag{25}$$
$$+ O\left(\log y \min\left(T, \frac{y}{\langle y\rangle}\right)\right) + O\left(\log 2T \min\left(T, \frac{1}{\log y}\right)\right)$$

for $y,\ T > 1$, where ρ denotes a complex zero of $\zeta(s)$ and $\langle y\rangle$ denotes the distance from y to the nearest prime power other than y itself. This formula is unconditional. Assuming RH and using (25) for the inner sum occurring in the first two terms of the right-hand side of (24), the contribution from the first term of the right-hand side of (25) is

$$\begin{aligned}\frac{T}{2\pi}\left\{x^{-2}\sum_{n\le x} n\Lambda(n)^2 + x^2\sum_{n>x}\frac{\Lambda(n)^2}{n^3}\right\} &= \frac{T}{2\pi}\left\{x^{-2}\left(\frac{x^2}{2}\log x - \frac{x^2}{4} + O(x^{\frac{3}{2}}\log^3 2x)\right)\right.\\ &\left. + x^2\left(\frac{\log x}{2x^2} + \frac{1}{4x^2} + O(x^{-\frac{5}{2}}\log^3 2x)\right)\right\}\\ &= \frac{T}{2\pi}\log x + O(Tx^{-\frac{1}{2}}\log^3 2x). \end{aligned}\tag{26}$$

Here we have calculated the sums from

$$\sum_{n\le x}\Lambda(n) = x + O(x^{\frac{1}{2}}\log^2 2x) \tag{27}$$

which is an expression of the prime number theorem under RH. The first error term in (25) contributes

$$\begin{aligned}&\ll x^{-2}\sum_{n\le x} n^2\Lambda(n)\log 2nT\log\log 3n + x^2\sum_{n>x}\frac{\Lambda(n)}{n^2}\log 2nT\log\log 3n\\ &\ll x\log 2xT\log\log 3x. \end{aligned}\tag{28}$$

The second error term in (25) contributes

$$\begin{aligned}&\ll \frac{1}{x^2}\sum_{n\le x} n\Lambda(n)\log n\min(T,n) + x^2\sum_{n>x}\frac{\Lambda(n)}{n^3}\log n\min(T,n)\\ &\ll \begin{cases} x\log 2x + \frac{x^2}{T}\log T & \text{if} \quad x\le T,\\ T\log x & \text{if} \quad x > T. \end{cases}\end{aligned}\tag{29}$$

Finally the last error term of (25) contributes

$$\ll \frac{1}{x^2}\sum_{n\le x} n\Lambda(n)\frac{\log 2T}{\log n} + x^2\sum_{n>x}\frac{\Lambda(n)}{n^3}\frac{\log 2T}{\log n} \ll \frac{\log T}{\log 2x}. \tag{30}$$

Combining the above we find that

$$F_{\zeta,\zeta}(x,T) = \frac{T\log^2 T}{2\pi x^2}\left(1+O\left(\frac{1}{\log T}\right)\right)+\frac{T}{2\pi}\log x \\ +O(x\log(xT)\ \log\log 3x)+O\left(\frac{T\log^3 2x}{x^{\frac{1}{2}}}\right) \tag{31}$$

as $T\to\infty$. This gives an asymptotic result for $1\le x = o\big(\frac{T}{\log\log T}\big)$.

5 Preliminaries Concerning the Relative Maxima of $|\zeta(\frac{1}{2}+it)|$

We shall be concerned with the function

$$Z_1(s) := \zeta'(s) - \frac{1}{2}\frac{\chi'}{\chi}(s)\zeta(s) \tag{32}$$

which was introduced by Conrey and Ghosh [1]. We easily see the relation

$$\left|Z_1\left(\frac{1}{2}+it\right)\right| = |Z'(t)| \tag{33}$$

with Hardy's Z-function

$$Z(t) = \left(\chi\left(\frac{1}{2}+it\right)\right)^{-\frac{1}{2}}\zeta\left(\frac{1}{2}+it\right), \tag{34}$$

which is real for real t and satisfies $|Z(t)| = |\zeta(\frac{1}{2}+it)|$. Taking derivatives of both sides of (34), and observing that $\frac{\chi'}{\chi}(\frac{1}{2}+it)$ is a real number, we see that

$$\frac{Z'}{Z}(t) = \Im\frac{\zeta'}{\zeta}\left(\frac{1}{2}-it\right). \tag{35}$$

Taking imaginary parts in the partial fraction formula (from Chap. 12 of [2])

$$\frac{\zeta'}{\zeta}(s) = -1-\frac{\gamma_e}{2}+\log 2\pi-\frac{1}{s-1}-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right)+\sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) \tag{36}$$

(here γ_e denotes Euler's constant), we have

$$\Im\frac{\zeta'}{\zeta}\left(\frac{1}{2}-it\right) = -\frac{t}{\frac{1}{4}+t^2}-\frac{1}{2}\Im\frac{\Gamma'}{\Gamma}\left(\frac{5}{4}-\frac{it}{2}\right)+\Im\sum_\rho\frac{1}{\frac{1}{2}-it-\rho}. \tag{37}$$

Now we assume RH, and use (15), to obtain

$$\frac{Z'}{Z}(t) = \Im \frac{\zeta'}{\zeta}\left(\frac{1}{2} - it\right) = \frac{\pi}{4} - O\left(\frac{1}{t}\right) + \sum_{\gamma>0} \frac{2t}{t^2-\gamma^2}. \tag{38}$$

This formula reveals that as $\frac{1}{2} + it$ crawls up on the critical line from a zero of $\zeta(s)$ to the next zero, there will be only one point where $Z'(t) = 0$. Hence we obtain that on RH, the zeros of $Z'(t)$ are interlaced with the zeros of $Z(t)$ (the case of possible multiple zeros of $Z(t)$ can also be included in this statement), and $|\zeta(\frac{1}{2} + it)|$ has exactly one maximum between consecutive zeta zeros in the upper half-plane (with the symmetrical configuration in the lower half-plane).

We shall need to know the zeros and poles of $Z_1(s)$, but before that let us consider the same for $\chi(s)$ and its derivative. From (12), since the Γ-function never vanishes, it is clear that the zeros of $\chi(s)$ can only be those zeros of $\sin(\frac{\pi s}{2})$ which are not cancelled by the poles of $\Gamma(1-s)$, i.e.

$$\{s;\ \chi(s) = 0\} = \{0, -2, -4, \ldots\}. \tag{39}$$

Similarly, we see

$$\{s;\ s \text{ is a pole of } \chi(s)\} = \{1, 3, 5, \ldots\}, \tag{40}$$

and that all zeros and poles of $\chi(s)$ are simple. By Rolle's theorem there are the real zeros of $\chi'(s)$ between the zeros of $\chi(s)$. A little computational examining using (14) reveals that

$$\begin{aligned} \{s \in \mathbb{R};\ \chi'(s) = 0\} \ &= \ \{\kappa_\ell, 1-\kappa_\ell; \ell \in \mathbb{Z}^+\},\ \kappa_1 \in \left(-\frac{1}{2}, 0\right),\ \kappa_2 \in (-3,-2), \\ &\kappa_3 \ \in \ (-5,-4),\ \kappa_\ell \in (-2\ell, -2\ell+1) \text{ for } \ell \geq 4, \\ &\kappa_\ell \to -2\ell \quad \text{as } \ \ell \to \infty. \end{aligned} \tag{41}$$

These zeros are all simple and the symmetry with respect to the point $\frac{1}{2}$ is due to

$$\frac{\chi'}{\chi}(s) = \frac{\chi'}{\chi}(1-s) \tag{42}$$

which is obvious from the functional equation of $\zeta(s)$. To look for non-real zeros of $\chi'(s)$, we use

$$\frac{\Gamma'}{\Gamma}(s) = -\frac{1}{s} - \gamma_e + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{s+n}\right), \quad (-s \notin \mathbb{N}) \tag{43}$$

in

$$\frac{\chi'}{\chi}(s) = \log\pi - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right), \tag{44}$$

to see that

$$\Im\,\frac{\chi'}{\chi}(\sigma+it)=\frac{t}{4}\sum_{n=0}^{\infty}\left(\frac{1}{\left(\frac{1-\sigma}{2}+n\right)^2+\left(\frac{t}{2}\right)^2}-\frac{1}{(\frac{\sigma}{2}+n)^2+\left(\frac{t}{2}\right)^2}\right) \tag{45}$$

except at the poles. Thus the zeros of $\chi'(s)$ can occur either when $t=0$ (and these have already been identified), or when $\sigma=\frac{1}{2}$. It is customary to write

$$\chi\left(\frac{1}{2}+it\right)=e^{-2i\vartheta(t)},\quad (\vartheta(t)\in\mathbb{R}), \tag{46}$$

where

$$\vartheta(t)=\Im\log\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)-\frac{t}{2}\log\pi=\frac{t}{2}\log\left(\frac{t}{2\pi}\right)-\frac{t}{2}-\frac{\pi}{8}+\frac{1}{48t}+\frac{7}{5760t^3}+\cdots \tag{47}$$

(see Sect. 6.5 of [4]). From (46), we see

$$\frac{\chi'}{\chi}\left(\frac{1}{2}+it\right)=-2\vartheta'(t)\in\mathbb{R}, \tag{48}$$

so that, by (43) and (44) we have

$$2\vartheta'(t)=-\Re\,\frac{\chi'}{\chi}\left(\frac{1}{2}+it\right)=-(\log\pi+\gamma_e)-\frac{1}{\frac{1}{4}+t^2}+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{\frac{1}{4}+n}{(\frac{1}{4}+n)^2+(\frac{t}{2})^2}\right). \tag{49}$$

This implies

$$2\vartheta''(t)=\sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}+n\right)\frac{t}{2}}{\left(\left(\frac{1}{4}+n\right)^2+\left(\frac{t}{2}\right)^2\right)^2}>0,\quad (t>0), \tag{50}$$

i.e. $\vartheta'(t)$ is strictly increasing for $t>0$. We find from (14) that

$$2\vartheta'(0)=-\frac{\chi'}{\chi}\left(\frac{1}{2}\right)=-\left(\gamma_e+\log 4+\frac{\pi}{2}+\log 2\pi\right)\approx -5.37\ldots, \tag{51}$$

and we know from Sect. 6.5 of [4] that

$$\vartheta'(t)>0,\quad (t\geq 10). \tag{52}$$

We deduce that

$$\chi'(s)=0,\ s\notin\mathbb{R}\implies s=\frac{1}{2}\pm i\tau,\text{ for a unique }\tau\in(0,10) \tag{53}$$

(τ must be near 2π, since from (47) we see $2\vartheta'(t)=\log(\frac{t}{2\pi})+O(\frac{1}{t^2})$).

Hall's paper [14] contains thorough information about the zeros and poles of $Z_1(s)$ (Hall denotes our $Z_1(s)$ by $F(s)$). We see that

$$\{s;\ s \text{ is a pole of } Z_1(s)\} = \{0, 1, 3, 5, 7, \ldots\} \tag{54}$$

(the pole at $s = 1$ is a double pole coming from the simple poles of $\zeta(s)$ and of $\chi(s)$ at $s = 1$, the simple pole at $s = 0$ arises from $\chi(0) = 0$, and the simple poles at $s = 3, 5, 7, \ldots$ come from the poles of $\chi(s)$ at these points). Hall, assuming RH, showed that all non-real (which will be termed as non-trivial) zeros of $Z_1(s)$ are on the critical line, so that, assuming RH, the non-real points where $Z_1(s)$ vanishes are the relative maxima of $|\zeta(\frac{1}{2} + it)|$, and the multiple zeta zeros if ever these exist. Some information about these have already been given in the beginning of this section. Hall also proved assuming RH that the number of the zeros of $Z_1(s)$ (counted according to multiplicity) with ordinates in $(0, T]$, $N_{Z_1}(T)$, is

$$N_{Z_1}(T) = N_\zeta(T) - \frac{1}{2}\operatorname{sgn}\frac{Z'}{Z}(t) + \frac{3}{2}, \tag{55}$$

provided that T is not the ordinate of a zero of $\zeta(s)$ or of $Z_1(s)$. There are real zeros of $Z_1(s)$, all of them simple zeros. These may be termed as the trivial zeros of $Z_1(s)$. The trivial zeros are placed symmetrically with respect to the point $\frac{1}{2}$ because of the functional equation

$$Z_1(s) = -\chi(s)Z_1(1 - s), \tag{56}$$

which follows from (32) and (11). We have

$$\{s;\ s \text{ is a trivial zero of } Z_1(s)\} = \{z_0, z_\ell, 1 - z_\ell; \ell \in \mathbb{Z}^+\},$$
$$z_0 = \frac{1}{2},\ z_1 \in (3, 5), z_2 \in (5, 7),\ z_\ell \in (2\ell + 2, 2\ell + 3)$$
$$\text{for } \ell \geq 3. \tag{57}$$

Since $Z_1(s)$ doesn't have an associated Dirichlet series, we will need a Dirichlet polynomial which approximates $\dfrac{Z_1'}{Z_1}(s)$ in $\sigma > 1,\ |t| \geq t_0$. From (32) we have

$$\frac{Z_1'}{Z_1}(s) = \frac{\frac{\zeta'}{\zeta}(s) - \frac{2}{\frac{\chi'}{\chi}(s)}\frac{\zeta''}{\zeta}(s) + \frac{\left(\frac{\chi'}{\chi}(s)\right)'}{\frac{\chi'}{\chi}(s)}}{1 - \frac{2}{\frac{\chi'}{\chi}(s)}\frac{\zeta'}{\zeta}(s)}. \tag{58}$$

Here we use the estimates (16), (17), and

$$\frac{\zeta^{(j)}}{\zeta}(s) \ll (\log\log|t|)^j, \quad (j = 1, 2;\ \sigma \geq 1,\ |t| \geq t_0), \tag{59}$$

which depends on RH and follows from Corollary 13.14 of [19] by a standard application of Cauchy's estimate, so that we can write

$$\frac{Z_1'}{Z_1}(s) = \frac{\frac{\zeta'}{\zeta}(s) + \frac{2}{\log\frac{|t|}{2\pi}}\frac{\zeta''}{\zeta}(s)}{1 + \frac{2}{\log\frac{|t|}{2\pi}}\frac{\zeta'}{\zeta}(s)} + O\left(\frac{1}{|t|\log|t|}\right). \tag{60}$$

Expanding the denominator as geometric series, we see that

$$\frac{1}{1 + \frac{2}{\log\frac{|t|}{2\pi}}\frac{\zeta'}{\zeta}(s)} = \sum_{k=0}^{\lfloor\frac{\log|t|}{\log\log|t|}\rfloor}\left(-\frac{2}{\log\frac{|t|}{2\pi}}\frac{\zeta'}{\zeta}(s)\right)^k + O\left(|t|^{-1}\exp\left(\frac{C_1\log|t|\log\log\log|t|}{\log\log|t|}\right)\right). \tag{61}$$

Now, for $k \in \mathbb{N}$ and $\sigma > 1$, we write

$$\left(-\frac{\zeta'}{\zeta}(s)\right)^k =: \sum_{n=1}^{\infty}\frac{\Lambda^{(k)}(n)}{n^s}, \tag{62}$$

$$\left(-\frac{\zeta'}{\zeta}(s)\right)^k\frac{\zeta''}{\zeta}(s) =: \sum_{n=1}^{\infty}\frac{\lambda^{(k)}(n)}{n^s}. \tag{63}$$

Using (61)–(63) in (60) gives the following Dirichlet polynomial approximation (the series begins with $m = 2$ since $\Lambda^{*(k+1)}(1) = \lambda^{*(k)}(1) = 0$).

Proposition 1 *Assume RH. In the region* $1 < \sigma \le \sigma_0$, $|t| \ge t_0$, *we have*

$$\frac{Z_1'}{Z_1}(s) = \sum_{m=2}^{\infty}\frac{1}{m^s}\sum_{k=0}^{\lfloor\frac{\log|t|}{\log\log|t|}\rfloor}\left(\frac{2}{\log\frac{|t|}{2\pi}}\right)^k\left(\frac{2}{\log\frac{|t|}{2\pi}}\lambda^{(k)}(m) - \Lambda^{(k+1)}(m)\right) + O\left(|t|^{-1}\exp\left(\frac{C_1\log|t|\log\log\log|t|}{\log\log|t|}\right)\right). \tag{64}$$

Remark We were led to the approximation (64) because the idea of replacing $-\frac{\chi'}{\chi}(s)$ by $\log\frac{T}{2\pi}$ in (32) and then developing an approximating Dirichlet series, as was done in [1], ends up giving a weaker (valid only for x up to $T^{\frac{2}{3}-\epsilon}$) result than Theorem 1 in Sect. 7. Unconditionally, we have $\frac{\zeta'}{\zeta}(s) \ll_j (\log t)^{\frac{2}{3}}(\log\log|t|)^{\frac{1}{3}}$, $(\sigma \ge 1,\ |t| \ge t_0)$ at our disposal (see [21], Sect. 6.19), causing the error term in (64) to become $O(|t|^{-\frac{1}{3}+\epsilon})$, and this in turn also leads to a smaller range in Theorem 1. Anyhow, results of the type we are pursuing make more sense when RH is assumed.

6 Preliminaries Concerning the Iterated Convolutions of the von Mangoldt Function

We shall need in Sects. 7 and 8 some properties of the arithmetic functions $\Lambda^{(k)}(n)$ and $\lambda^{(k)}(n)$. We note that the article [5] by Farmer et al. contains some related results. First, proceeding inductively and denoting Dirichlet convolution by $*$, we see the trivial bound, for $k \geq 1$,

$$\Lambda^{(k)}(n) = \Lambda^{(k-1)} * \Lambda(n) = \sum_{d|n} \Lambda(d)\Lambda^{(k-1)}\left(\frac{n}{d}\right) \leq (\log n)^{k-1} \sum_{d|n} \Lambda(d) = (\log n)^k. \tag{65}$$

Next, using (65), we see that

$$\lambda^{(k)}(n) \leq (\log n)^{k+2}, \qquad (k = 0, 1, 2, \ldots), \tag{66}$$

by dint of

$$\lambda^{(k)} = \Lambda^{(k)} * \Lambda_2, \text{ where } \frac{\zeta''}{\zeta}(s) =: \sum_{n=1}^{\infty} \frac{\Lambda_2(n)}{n^s} \ (\Re\, s > 1), \text{ and } \sum_{d|n} \Lambda_2(d) = (\log n)^2. \tag{67}$$

Note that by comparing coefficients in $\frac{\zeta''}{\zeta}(s) = \left(\frac{\zeta'}{\zeta}(s)\right)' + \left(\frac{\zeta'}{\zeta}(s)\right)^2$ we have

$$\lambda^{(0)}(n) = \Lambda_2(n) = \Lambda(n)\log n + \Lambda^{(2)}(n) = \Lambda(n)\log n + \sum_{d|n} \Lambda(d)\Lambda\left(\frac{n}{d}\right). \tag{68}$$

It is easy to calculate that for $a \in \mathbb{Z}^+$ and p a prime number,

$$\Lambda_2(p^a) = (2a-1)(\log p)^2 \tag{69}$$

$$\Lambda^{(k)}(p^a) = \binom{a-1}{k-1}(\log p)^k, \tag{70}$$

$$\lambda^{(k)}(p^a) = \left[2\binom{a}{k+1} - \binom{a-1}{k}\right](\log p)^{k+2}. \tag{71}$$

Here we are employing a convention that for u, v integers which are ≥ -1,

$$\binom{r}{s} := \begin{cases} 1 & \text{if} \quad r = s, \\ 0 & \text{if} \quad s = -1 \text{ and } r \geq 0, \\ 0 & \text{if} \quad r < s, \\ \dfrac{r!}{s!(r-s)!} & \text{otherwise.} \end{cases} \tag{72}$$

Now we give two lemmas which will be used in Sect. 7.

Lemma 1 *Let $k \in \mathbb{Z}^+$. We have, for $N \geq 1$,*

$$\sum_{n \leq \frac{N}{2}} \frac{\Lambda^{(k)}(N-n)}{n} \ll (\log 2N)^k \frac{(\log\log 3N+7)_k}{(k-1)!}. \tag{73}$$

The same bound holds with $\Lambda^{(k)}(N+n)$ in place of $\Lambda^{(k)}(N-n)$. Here

$$(K)_k := \prod_{j=0}^{k-1}(K+j), \tag{74}$$

with the convention $(K)_0 := 1$.
The $k = 1$ case of this lemma was shown by Gonek [13] within his proof of (25).

Proof Let

$$S(x) := \left| \left\{ N - x < n \leq N : \Lambda^{(k)}(n) \neq 0 \right\} \right|.$$

By (65) we can write

$$\sum_{n \leq \frac{N}{2}} \frac{\Lambda^{(k)}(N-n)}{n} \leq (\log 2N)^k \int_{1-}^{\frac{N}{2}} \frac{dS(x)}{x} = (\log 2N)^k \left\{ \frac{S(x)}{x}\Big|_{1-}^{\frac{N}{2}} + \int_{1-}^{\frac{N}{2}} \frac{S(x)}{x^2} dx \right\}$$

We now need an upper bound for $S(x)$. To this end we quote the following result of Tudesq [22]:

$$\sum_{\substack{y < n \leq y+x \\ \omega(n)=j}} 1 \ll \frac{x}{\log 2x} \frac{(\log\log 3x+6)_{j-1}}{(j-1)!},$$

where $1 \leq x \leq y$, $j \in \mathbb{Z}^+$ and the constant implied by $\ll$ is absolute. Employing this we have

$$S(x) \leq \sum_{j=1}^{k} \sum_{\substack{N-x < n \leq N \\ \omega(n)=j}} 1 \ll \frac{x}{\log 2x} \sum_{j=1}^{k} \frac{(\log\log 3x+6)_{j-1}}{(j-1)!} = \frac{x}{\log 2x} \frac{(\log\log 3x+7)_{k-1}}{(k-1)!},$$

since

$$\sum_{j=0}^{k} \frac{(r)_j}{j!} = \frac{(r+1)_k}{k!}, \quad (k \in \mathbb{N},\ r \in \mathbb{R}). \tag{75}$$

This upper bound for $S(x)$ gives the claimed result. The sum having $\Lambda^{(k)}(N+n)$ in its summand can be treated in the same manner. □

By the same line of reasoning we have

Lemma 2 *Let $k \in \mathbb{N}$. We have for $N \geq 1$*

$$\sum_{n \leq \frac{N}{2}} \frac{\lambda^{(k)}(N-n)}{n} \ll (\log 2N)^{k+2} \frac{(\log\log 3N+7)_{k+2}}{(k+1)!}. \tag{76}$$

The same bound holds with $\lambda^{(k)}(N+n)$ in place of $\lambda^{(k)}(N-n)$.

The rest of this section will be used in Sect. 8. We define

$$\sum_{n=1}^{\infty} \frac{\Delta(k,\iota;n)}{n^s} := \left(-\frac{\zeta'}{\zeta}(s)\right)^k \left(\frac{\zeta''}{\zeta}(s)\right)^{\iota}, \quad \iota = 0 \text{ or } 1 \tag{77}$$

so that (cf. (62)–(63))

$$\Delta(k,0;n) = \Lambda^{(k)}(n), \quad \Delta(k,1;n) = \lambda^{(k)}(n). \tag{78}$$

We are going to study the sum $\sum_{n \leq x} \Delta(k_1,\iota_1,n)\Delta(k_2,\iota_2,n)$. The result will be Proposition 2, and its corollary will be needed in Sect. 8. We begin with the following Lemmas.

Lemma 3 *For prime p and $k \in \mathbb{N}$, we have*

$$\Delta(k,\iota;pn) = \begin{cases} k(\log p)\Delta(k-1,\iota;n) \\ \quad +[\iota=1](\log p)\left[2\Delta(k+1,0;n)+(\log p)\Delta(k,0;n)\right] & \text{if } (p,n)=1, \\ O\left((k+1)(\log p)(\log pn)^{k+2\iota-1}\right) & \text{if } p|n. \end{cases} \tag{79}$$

Proof The $k = 0$ case can be verified easily, so let $k \geq 1$. First take $\iota = 0$. For $(p,m) = 1$, by (78) and (70) we have

$$\begin{aligned} \Delta(k,0;p^a m) = \Lambda^{(k)}(p^a m) &= \sum_{r=1}^{\min(a,k)} \binom{k}{r} \Lambda^{(r)}(p^a)\Lambda^{(k-r)}(m) \\ &= \sum_{r=1}^{k} \binom{k}{r}\binom{a-1}{r-1} (\log p)^r \Lambda^{(k-r)}(m). \end{aligned} \tag{80}$$

With $a = 1$ this reads, for $(p,n) = 1$,

$$\Delta(k,0;pn) = \Lambda^{(k)}(pn) = k(\log p)\Lambda^{(k-1)}(n) = k(\log p)\Delta(k-1,0;n)$$

which is the statement of (79) for this case. Using $\binom{a-1}{r-1} \le \frac{a^{r-1}}{(r-1)!}$ in (80), by (65) we have

$$\begin{aligned}\Delta(k,0;p^a m) = \Lambda^{(k)}(p^a m) &\le \sum_{r=1}^{k} \frac{k}{r}\binom{k-1}{r-1}\frac{a^{r-1}}{(r-1)!}(\log p)^r (\log m)^{k-r} \\ &\le k(\log p)\sum_{r=1}^{k}\binom{k-1}{r-1}(\log p^a)^{r-1}(\log m)^{(k-1)-(r-1)} \\ &= k(\log p)(\log p^a m)^{k-1}.\end{aligned}$$

This completes the case $\iota = 0$ and $p \mid n$.

Now we consider the case $\iota = 1$. For $(p,n) = 1$, we have

$$\begin{aligned}\Delta(k,1;pn) = \lambda^{(k)}(pn) &= \sum_{d\mid pn} \Lambda^{(k)}\left(\frac{pn}{d}\right)\Lambda_2(d) \\ &= \sum_{d\mid n}\Lambda^{(k)}\left(\frac{pn}{d}\right)\Lambda_2(d) + \sum_{d\mid n}\Lambda^{(k)}\left(\frac{n}{d}\right)\Lambda_2(pd)\end{aligned}$$

In the last member of this equation, applying the already proved $\iota = 0$ case to $\Lambda^{(k)}$ in the first sum, and using (68) in the second sum we obtain

$$\begin{aligned}\Delta(k,\iota;pn) = k(\log p)\sum_{d\mid n}\Lambda^{(k-1)}\left(\frac{n}{d}\right)\Lambda_2(d) + 2(\log p)\sum_{\substack{d\mid n\\ d>1}}\Lambda^{(k)}\left(\frac{n}{d}\right)\Lambda(d) \\ +(\log p)^2\Lambda^{(k)}(n),\end{aligned}$$

which gives the statement of (79) for this case.

For the last case with $p \mid n$, let $n = p^r n'$, $(n',p) = 1$, $r \in \mathbb{Z}^+$. Observe that

$$\begin{aligned}\Delta(k,1;pn) = \lambda^{(k)}(pn) &= \sum_{d\mid pn}\Lambda^{(k)}\left(\frac{pn}{d}\right)\Lambda_2(d) \\ &= \sum_{d\mid n}\Lambda^{(k)}\left(\frac{pn}{d}\right)\Lambda_2(d) + \sum_{\substack{d'\mid n'\\ d'>1}}\Lambda^{(k)}\left(\frac{n'}{d'}\right)\Lambda_2(p^{r+1}d') + \Lambda^{(k)}(n')\Lambda_2(p^{r+1}).\end{aligned}$$

Using (65), (69),

$$\Lambda_2(p^{r+1}d') = 2(\log p)\Lambda(d'), \quad (d' > 1,\ (d',p) = 1) \tag{81}$$

which follows from (68), and the case $\iota = 0$, we obtain

$$\Delta(k, 1; pn) \leq k(\log p)(\log pn)^{k-1} \sum_{d|n} \Lambda_2(d) + 2(\log p)(\log n')^k \sum_{d'|n'} \Lambda(d') + (2r+1)(\log p)^2 (\log n')^k.$$

Now the assertion in the case $\iota = 1$ and $(p, n) > 1$ is immediately seen. □

We now begin examining the sum $\sum_{n \leq \frac{x}{b}} \Delta(k, \iota; bn)$, where b is either 1 or a prime number $p \leq x$ (clearly, the sum is void if $b > x$), for $k \leq \frac{\log x}{\log 2}$ (if $k > \frac{\log x}{\log 2}$, then $\Delta(k, \iota; bn) = 0$). The results will form

Lemma 4 *Let b be 1 or any prime number $p \leq x$. For $1 + [b \neq 1] \leq k + \iota \leq \frac{\log x}{\log 2}$, there exists a constant $C_2 > 1$ such that*

$$\sum_{n \leq \frac{x}{b}} \Delta(k, \iota; bn) \leq \frac{C_2^k}{(k + 2\iota - 1)!} \frac{x}{b} (\log x)^{k+2\iota-1}. \tag{82}$$

Proof First we take $b = 1$. We have

$$\sum_{n \leq x} \Delta(k, \iota; n) \sim \frac{2^\iota x (\log x)^{k+2\iota-1}}{(k + 2\iota - 1)!}, \ (x \to \infty, \text{ any } \text{ fixed } k \in \mathbb{N}, \iota \in \{0, 1\}, k + \iota \geq 1) \tag{83}$$

as a generalization of the prime number theorem, since

$$\text{Res}_{s=1} \left[\left(-\frac{\zeta'}{\zeta}(s) \right)^k \left(\frac{\zeta''}{\zeta}(s) \right)^\iota \frac{x^s}{s} \right] \sim \frac{2^\iota x (\log x)^{k+2\iota-1}}{(k + 2\iota - 1)!}, \ (k \in \mathbb{N}, \iota \in \{0, 1\}, k + \iota \geq 1) \tag{84}$$

(the details of the proof of (83) are similar to the zeta-function based proof of the prime number theorem as in [2]). However, we will also need to work with k not fixed. We want to show that

$$\sum_{n \leq x} \Delta(k, \iota; n) \leq \frac{C_3^k}{(k + 2\iota - 1)!} x (\log x)^{k+2\iota-1} \tag{85}$$

with some absolute constant $C_3 > 1$ by doing induction on k. The induction basis cases are covered in (83). Starting from the definition in (77) we see that

$$\sum_{n\le x}\Delta(k+1,\iota;n)=\sum_{n\le x}\sum_{d|n}\Delta\left(k,\iota;\frac{n}{d}\right)\Lambda(d)=\sum_{d\le x}\Lambda(d)\sum_{e\le\frac{x}{d}}\Delta(k,\iota;e)$$

$$\le\frac{C_3^k x}{(k+2\iota-1)!}\sum_{d\le x}\frac{\Lambda(d)}{d}\left(\log\frac{x}{d}\right)^{k+2\iota-1}$$

$$\le\frac{C_3^{k+1}}{(k+2\iota)!}x(\log x)^{k+2\iota},\tag{86}$$

where we have used

$$\sum_{n\le x}\frac{\Lambda(n)}{n}\left(\log\frac{x}{n}\right)^j=\frac{(\log x)^{j+1}}{j+1}+O\left((\log x)^j\right),\quad(j\in\mathbb{N})\tag{87}$$

which can be seen from Mertens's classical result via partial summation (this result is included in Lemma 5.5 of [5]). This proves (85).

Next, taking $b=p$ a prime number, we examine the sum $\sum_{n\le\frac{x}{p}}\Delta(k,\iota;pn)$. Let ℓ be the unique integer such that $x\in[p^\ell,p^{\ell+1})$. First we examine the cases $k+\iota=1$ which turn out to be special. For $(k,\iota)=(1,0)$, the sum is $\sum_{n\le\frac{x}{p}}\Lambda(pn)$. The non-zero summands are from $n=1,p,p^2,\dots,p^{\ell-1}$, and the sum has value $\ell\log p\in(\frac{1}{2}\log x,\log x]$, so that

$$\sum_{n\le\frac{x}{p}}\Delta(1,0;pn)=\sum_{n\le\frac{x}{p}}\Lambda(pn)\le\log x.\tag{88}$$

For $(k,\iota)=(0,1)$, the sum is $\sum_{n\le\frac{x}{p}}\Lambda_2(pn)$. The non-zero summands are from $n=1,p,p^2,\dots,p^{\ell-1}$ with total contribution $\ell^2(\log p)^2\in(\frac{1}{4}(\log x)^2,(\log x)^2]$, and from $n=p^rp_1^{r_1}\le\dfrac{x}{p}$ with a prime $p_1\ne p$ and $r\ge0$, $r_1\ge1$. By

$$\Lambda_2(p_1^{a_1}p_2^{a_2})=2\log p_1\log p_2,\quad(a_1,a_2\ge1),\tag{89}$$

which is seen from (68), and

$$\theta(x):=\sum_{\substack{p\le x\\ p:\text{ prime}}}\log p\le C_4x,\quad(x>0)\tag{90}$$

(with $C_4 = 1 + \frac{1}{36260}$ according to Dusart [3]), we have

$$\begin{aligned}\sum_{\substack{p^r p_1^{r_1} \le \frac{x}{p} \\ r\ge 0, r_1 \ge 1}} \Lambda_2(p^{r+1}p_1^{r_1}) &\le 2\log p \sum_{r\ge 0}\sum_{r_1\ge 1} \sum_{p_1 \le \left(\frac{x}{p^{r+1}}\right)^{\frac{1}{r_1}}} \log p_1 \\ &\le 2C_4 \log p \sum_{r\ge 0}\sum_{r_1\ge 1} \left(\frac{x}{p^{r+1}}\right)^{\frac{1}{r_1}} \\ &\le C_5 \log p \sum_{r\ge 0} \frac{x}{p^{r+1}} \le C_6 \frac{x}{p}\log p, \end{aligned} \tag{91}$$

where we have made use of $\left(\frac{x}{p^{r+1}}\right)^{\frac{1}{r_1}} \ge 2$ for non-void sums so that r_1 can be at most $\frac{\log \frac{x}{p^{r+1}}}{\log 2}$. We have chosen to use the majorization (90) because if we try to evaluate all of the sums asymptotically we will get quantities containing the factor $\log \frac{x}{p}$ and this will necessitate examining the results according to the size of p relative to x and thus further complicate the matter. So, we express the result for this case as

$$\sum_{n\le \frac{x}{p}} \Delta(0,1;pn) = \sum_{n\le \frac{x}{p}} \Lambda_2(pn) \le C_7 \frac{x}{p}\log p + (\log x)^2. \tag{92}$$

We wish to obtain an inequality of the type (85) by induction on $k+\iota$, but from (88) and (92) we see that in the $k+\iota = 1$ cases the power of $\log x$ won't match the power in (85) if $\frac{x}{\log x} = o(p)$, $p \le x$. So, for the induction basis we take $k+\iota = 2$ which has the two cases $(k,\iota) = (2,0)$ and $(k,\iota) = (1,1)$.

For $(k,\iota) = (2,0)$, the sum is $\sum_{n\le \frac{x}{p}} \Lambda^{(2)}(pn)$, differing from $\sum_{n\le \frac{x}{p}} \Lambda_2(pn)$ only in the contribution from $pn = p, p^2, \ldots, p^\ell$. This contribution, $\frac{\ell(\ell-1)}{2}(\log p)^2$, vanishes for $p > \sqrt{x}$. Hence, from (92) we have

$$\sum_{n\le \frac{x}{p}} \Delta(2,0;pn) = \sum_{n\le \frac{x}{p}} \Lambda^{(2)}(pn) \le C_8 \frac{x}{p}\log x. \tag{93}$$

Now take $(k,\iota) = (1,1)$ in which case the sum is $\sum_{n\le \frac{x}{p}} \lambda^{(1)}(pn)$. The non-zero summands are from $n = p, p^2, \ldots, p^{\ell-1}$ with total contribution $\frac{\ell(\ell-1)(2\ell-1)}{6}(\log p)^3$ by (71), and from $n = p^r p_1^{r_1}$, $(r \ge 0, r_1 \ge 1)$ and $n = p^r p_1^{r_1} p_2^{r_2}$, $(r \ge 0, r_1, r_2 \ge 1)$ where p, p_1 and p_2 are distinct primes. We have, by (67), (69) and (89),

$$\lambda^{(1)}(p^{r+1}p_1^{r_1}) = (4r+1)(\log p)^2(\log p_1) + (4r_1 - 3)(\log p_1)^2(\log p) \tag{94}$$

$$\lambda^{(1)}(p^{r+1}p_1^{r_1}p_2^{r_2}) = 6(\log p)(\log p_1)(\log p_2). \tag{95}$$

Then, estimating similarly to (91), we have

$$\begin{aligned}
\sum_{\substack{p^r p_1^{r_1} \le \frac{x}{p} \\ r\ge 0, r_1 \ge 1}} \lambda^{(1)}(p^{r+1}p_1^{r_1}) \le\ & (\log p)^2 \sum_{r\ge 0}(4r+1)\sum_{r\ge 1}\sum_{p_1 \le \left(\frac{x}{p^{r+1}}\right)^{\frac{1}{r_1}}} \log p_1 \\
& + (\log p)\sum_{r\ge 0}\sum_{r_1\ge 1}(4r_1-3)\sum_{p_1 \le \left(\frac{x}{p^{r+1}}\right)^{\frac{1}{r_1}}} (\log p_1)^2 \\
\le\ & C_4(\log p)^2 \sum_{r\ge 0}(4r+1)\sum_{r\ge 1}\left(\frac{x}{p^{r+1}}\right)^{\frac{1}{r_1}} \\
& + C_4(\log p)\sum_{r\ge 0}\sum_{r_1\ge 1}(4r_1-3)\left(\frac{x}{p^{r+1}}\right)^{\frac{1}{r_1}} \log\left(\frac{x}{p^{r+1}}\right)^{\frac{1}{r_1}} \\
\le\ & C_9(\log p)^2 \sum_{r\ge 0}(4r+1)\frac{x}{p^{r+1}} \\
& + C_{10}(\log p)(\log x)\sum_{r\ge 0}\sum_{r_1\ge 1}\frac{4r_1-3}{r_1}\left(\frac{x}{p^{r+1}}\right)^{\frac{1}{r_1}} \\
\le\ & C_{11}\frac{x}{p}(\log p)(\log x),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\substack{p^r p_1^{r_1} p_2^{r_2} \le \frac{x}{p} \\ r\ge 0,\, r_1, r_2 \ge 1}} \lambda^{(1)}(p^{r+1}p_1^{r_1}p_2^{r_2}) \le\ & 3(\log p)\sum_{\substack{r\ge 0 \\ r_1\ge 1}}\sum_{p_1 \le \left(\frac{x}{2p^{r+1}}\right)^{\frac{1}{r_1}}} \log p_1 \sum_{r_2\ge 1}\sum_{p_2 \le \left(\frac{x}{p^{r+1}p_1^{r_1}}\right)^{\frac{1}{r_2}}} \log p_2 \\
\le\ & C_{12}(\log p)\sum_{\substack{r\ge 0 \\ r_1\ge 1}}\sum_{p_1 \le \left(\frac{x}{2p^{r+1}}\right)^{\frac{1}{r_1}}} \log p_1 \sum_{r_2\ge 1}\left(\frac{x}{p^{r+1}p_1^{r_1}}\right)^{\frac{1}{r_2}} \\
\le\ & C_{13}x(\log p)\sum_{r\ge 0}\frac{1}{p^{r+1}}\sum_{r_1\ge 1}\sum_{p_1 \le \left(\frac{x}{2p^{r+1}}\right)^{\frac{1}{r_1}}} \frac{\log p_1}{p_1^{r_1}} \\
\le\ & C_{14}x(\log p)\sum_{r\ge 0}\frac{1}{p^{r+1}}\sum_{p_1 \le \frac{x}{2p^{r+1}}} (\log p_1)\sum_{r_1\ge 1}\frac{1}{p_1^{r_1}} \\
\le\ & C_{15}\frac{x}{p}(\log p)(\log x).
\end{aligned}$$

Putting together these calculations, we have for the case $(k,\iota)=(1,1)$,

$$\sum_{n\le\frac{x}{p}}\Delta(1,1;pn)=\sum_{n\le\frac{x}{p}}\lambda^{(1)}(pn)\le C_{16}\frac{x}{p}(\log x)^2. \tag{96}$$

Formulas (93) and (96) are of the same type as (85), and this furnishes the induction basis for establishing

$$\sum_{n\le\frac{x}{p}}\Delta(k,\iota;pn)\le\frac{C_{17}^k}{(k+2\iota-1)!}\frac{x}{p}(\log x)^{k+2\iota-1}. \tag{97}$$

with some absolute constant $C_{17}>1$. Letting $pn=de$, we have

$$\begin{aligned}\sum_{n\le\frac{x}{p}}\Delta(k+1,\iota;pn)&=\sum_{n\le\frac{x}{p}}\sum_{d|pn}\Delta\left(k,\iota;\frac{pn}{d}\right)\Lambda(d)\\&=\sum_{\substack{d\le\frac{x}{p}\\(d,p)=1}}\Lambda(d)\sum_{\substack{e\le\frac{x}{d}\\p|e}}\Delta(k,\iota;e)+\sum_{\substack{d\le x\\p|d}}\Lambda(d)\sum_{e\le\frac{x}{d}}\Delta(k,\iota;e)\\&=\sum_{\substack{d\le\frac{x}{p}\\(d,p)=1}}\Lambda(d)\sum_{e'\le\frac{x}{pd}}\Delta(k,\iota;pe')+\sum_{d'\le\frac{x}{p}}\Lambda(pd')\sum_{e\le\frac{x}{pd'}}\Delta(k,\iota;e).\end{aligned} \tag{98}$$

The last two double sums can be majorized by using the induction hypothesis and (87). We have

$$\begin{aligned}\sum_{\substack{d\le\frac{x}{p}\\(d,p)=1}}\Lambda(d)\sum_{e'\le\frac{x}{pd}}\Delta(k,\iota;pe')&\le\frac{C_{17}^k}{(k+2\iota-1)!}\frac{x}{p}\sum_{d\le\frac{x}{p}}\frac{\Lambda(d)}{d}\left(\log\frac{x}{d}\right)^{k+2\iota-1}\\&\le\frac{C_{17}^k}{(k+2\iota-1)!}\frac{x}{p}\left(\frac{(\log x)^{j+2\iota}}{(k+2\iota)}+O\big((\log x)^{k+2\iota-1}\big)\right),\end{aligned}$$

and, since $\Lambda(pd')\le\Lambda(d')$,

$$\begin{aligned}\sum_{d'\le\frac{x}{p}}\Lambda(pd')\sum_{e\le\frac{x}{pd'}}\Delta(k,\iota;e)&\le\frac{C_{17}^k}{(k+2\iota-1)!}\frac{x}{p}\sum_{d'\le\frac{x}{p}}\frac{\Lambda(pd')}{d'}\left(\log\frac{x}{pd'}\right)^{k+2\iota-1}\\&\le\frac{C_{17}^k}{(k+2\iota-1)!}\frac{x}{p}\left(\frac{(\log x)^{k+2\iota}}{(k+2\iota)}+O\big((\log x)^{k+2\iota-1}\big)\right).\end{aligned}$$

Using these in (98), the induction step is completed. □

Next, we prove

Proposition 2 *For* $0 \le k_1, k_2 \le \frac{\log x}{\log 2}$, $\iota_1, \iota_2 = 0$ *or* 1, $k_1 + \iota_1$, $k_2 + \iota_2 \ge 1$, *we have*

$$\begin{aligned} S_{k_1,k_2,\iota_1,\iota_2}(x) &:= \sum_{n\le x} \Delta(k_1,\iota_1;n)\Delta(k_2,\iota_2;n) \\ &= \frac{P(k_1,k_2,\iota_1,\iota_2)}{(k_1+k_2+2\iota_1+2\iota_2-1)!} x(\log x)^{k_1+k_2+2\iota_1+2\iota_2-1} \\ &\quad + O\left(\frac{C_{18}^{k_1+k_2} x(\log x)^{k_1+k_2+2\iota_1+2\iota_2-2}}{(\max\{k_1,k_2\})!}\right), \end{aligned} \tag{99}$$

where

$$P(k_1,k_2,\iota_1,\iota_2) := \begin{cases} k_1! & \text{if} \quad \iota_1=\iota_2=0,\ k_1=k_2, \\ 2\cdot(\max\{k_1,k_2\})! & \text{if} \quad \iota_1-\iota_2=\pm 1,\ k_1-k_2=\mp 1, \\ (\max\{k_1,k_2\})! & \text{if} \quad \iota_1-\iota_2=\pm 1,\ k_1-k_2=\mp 2, \\ (k_1+2)!+4\cdot(k_1+1)!+2\cdot k_1! & \text{if} \quad \iota_1=\iota_2=1,\ k_1=k_2, \\ 2\cdot(\max\{k_1,k_2\}+1)! & \text{if} \quad \iota_1=\iota_2=1,\ k_1-k_2=\pm 1, \\ 0 & \text{otherwise.} \end{cases} \tag{100}$$

Proof We will do induction on $k_1 + k_2 + \iota_1 + \iota_2$, for which the least value is 2, and there are three cases of this to be considered for establishing the induction basis. We have

$$S_{1,1,0,0}(x) = \sum_{n\le x} \Lambda(n)^2 = x\log x + O(x), \tag{101}$$

$$\begin{aligned} S_{1,0,0,1}(x) = S_{0,1,1,0}(x) &= \sum_{n\le x} \Lambda(n)\Lambda_2(n) \\ &= \sum_{n\le x}\left[\Lambda(n)^2\log n + \Lambda(n)\sum_{d|n}\Lambda(d)\Lambda\Big(\frac{n}{d}\Big)\right] = x(\log x)^2 + O(x\log x), \end{aligned} \tag{102}$$

since the first term in brackets contributes $x(\log x)^2 + O(x\log x)$, and in the second term only proper prime power values of n contribute, and this contribution is $O(x^{\frac{1}{2}+\epsilon})$.
The final case for the induction basis is

$$\begin{aligned} S_{0,0,1,1}(x) &= \sum_{n\le x} \lambda^{(0)}(n)^2 = \sum_{n\le x} \Lambda_2(n)^2 \\ &= \sum_{n\le x}\left[\Lambda(n)^2(\log n)^2 + 2\Lambda(n)\log n\sum_{d|n}\Lambda(d)\Lambda\Big(\frac{n}{d}\Big) + \left(\sum_{d|n}\Lambda(d)\Lambda\Big(\frac{n}{d}\Big)\right)^2\right]. \end{aligned}$$

The first term in the summand leads to $x(\log x)^3 + O(x(\log x)^2)$. In the second term of the summand, only proper prime power n contribute, so that this term leads to $O(x^{\frac{1}{2}+\epsilon})$. The contribution of proper prime power n in the third term is bounded by the same error term. The third term is 0 if n has more than two distinct prime factors, so it remains to consider the contribution of n having exactly two distinct prime factors in the third term. The set of $n = p_1p_2 \leq x$, $p_1 < p_2$ contributes to the third term (we need not exclude the cases $p_1 = p_2$ since these give a negligible contribution, and we take all pairs of p_1 and p_2 and then halve the sum; we will use this setting also in other cases below)

$$\begin{aligned}
\frac{1}{2}\sum_{p_1p_2\leq x}(2\log p_1\log p_2)^2 &= 2\sum_{p_1\leq\frac{x}{2}}(\log p_1)^2\sum_{p_2\leq\frac{x}{p_1}}(\log p_2)^2\\
&= 2\sum_{p_1\leq\frac{x}{2}}(\log p_1)^2\left(\frac{x}{p_1}\log\left(\frac{x}{p_1}\right)+O\left(\frac{x}{p_1}\right)\right)\\
&= 2x\log x\sum_{p_1\leq\frac{x}{2}}\frac{(\log p_1)^2}{p_1}\\
&\quad -2x\sum_{p_1\leq\frac{x}{2}}\frac{(\log p_1)^3}{p_1}+O\Big(x(\log x)^2\Big)\\
&= 2x\log x\left(\frac{1}{2}(\log x)^2+O(\log x)\right)\\
&\quad -2x\left(\frac{1}{3}(\log x)^3+O\left((\log x)^2\right)\right)\\
&= \frac{1}{3}x(\log x)^3+O(x(\log x)^2).
\end{aligned}$$

The third term also gets contributions from $n = p_1^{r_1}p_2^{r_2}, p_1 \neq p_2, r_1 \geq 1, r_2 \geq 2$, but

$$\begin{aligned}
\sum_{r_1\geq1}\sum_{p_1^{r_1}\leq\frac{x}{4}}(\log p_1)^2\sum_{r_2\geq2}\sum_{p_2\leq\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}}(\log p_2)^2 &\ll \sum_{r_1\geq1}\sum_{p_1^{r_1}\leq\frac{x}{4}}(\log p_1)^2\sum_{r_2\geq2}\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}\log x\\
&\ll x^{\frac{1}{2}}\log x\sum_{r_1\geq1}\sum_{p_1^{r_1}\leq\frac{x}{4}}\frac{(\log p_1)^2}{p_1^{\frac{r_1}{2}}}\\
&= x^{\frac{1}{2}}\log x\sum_{p_1\leq\frac{x}{4}}\frac{(\log p_1)^2}{p_1^{\frac{1}{2}}}\ll x(\log x)^2.
\end{aligned}$$

Hence

$$S_{0,0,1,1}(x) = \frac{4}{3}x(\log x)^3 + O(x\log^2 x). \tag{103}$$

The formulas (101), (102) and (103) obey (99) and (the first, the second and the fourth cases, respectively, of) (100), and so the induction basis is established.

In dealing with the cases $k_1 + k_2 + \iota_1 + \iota_2 \geq 3$, we will first consider the cases when one of k_1 and k_2 is 0, say $k_1 = 0$ so that $\iota_1 = 1$. There are four such cases: $S_{0,1,1,1}(x)$, $S_{0,2,1,0}(x)$, $S_{0,k_2,1,0}$ with $k_2 \geq 3$, and $S_{0,k_2,1,1}(x)$ with $k_2 \geq 2$.

We begin by considering

$$S_{0,1,1,1}(x) = \sum_{n\leq x} \lambda^{(0)}(n)\lambda^{(1)}(n) = \sum_{n\leq x} \Lambda_2(n)(\Lambda * \Lambda_2)(n).$$

Here only those n which are either proper prime powers or which have exactly two distinct prime factors contribute. Using (68)–(71), the contribution of proper prime power n is

$$\sum_{p^a\leq x} \lambda^{(0)}(p^a)\lambda^{(1)}(p^a) = \sum_{p^a\leq x} (2a-1)(a-1)^2(\log p)^5 \ll x^{\frac{1}{2}+\epsilon}.$$

Using (94), the contribution of $n = p_1p_2$ terms is seen to be

$$\begin{aligned}
&\frac{1}{2}\sum_{p_1p_2\leq x} \lambda^{(0)}(p_1p_2)\lambda^{(1)}(p_1p_2)\\
&= \frac{1}{2}\sum_{p_1p_2\leq x} (2\log p_1\log p_2)\Big[\log p_1(\log p_2)^2 + (\log p_1)^2\log p_2\Big]\\
&= 2\sum_{p_1p_2\leq x} (\log p_1)^3(\log p_2)^2 = 2\sum_{p_1\leq \frac{x}{2}} (\log p_1)^3 \sum_{p_2<\frac{x}{p_1}} (\log p_2)^2\\
&= 2\sum_{p_1\leq \frac{x}{2}} (\log p_1)^3\Big(\frac{x}{p_1}\log\frac{x}{p_1} + O\Big(\frac{x}{p_1}\Big)\Big)\\
&= 2x\log x\sum_{p_1\leq \frac{x}{2}} \frac{(\log p_1)^3}{p_1} - 2x\sum_{p_1\leq \frac{x}{2}} \frac{(\log p_1)^4}{p_1} + O(x(\log x)^3)\\
&= \frac{1}{6}x(\log x)^4 + O(x(\log x)^3),
\end{aligned}$$

As for the contribution of those n having two distinct prime factors of which at least one is a proper prime power, we have

$$\sum_{\substack{p_1^{r_1}p_2^{r_2}\le x\\ r_1\ge 1,r_2\ge 2}} \lambda^{(0)}(p_1^{r_1}p_2^{r_2})\lambda^{(1)}(p_1^{r_1}p_2^{r_2}) \ll \sum_{\substack{p_1^{r_1}p_2^{r_2}\le x\\ r_1\ge 1,r_2\ge 2}} r_1(\log p_1)^3(\log p_2)^2 + r_2(\log p_1)^2(\log p_2)^3$$

$$\ll \sum_{r_1\ge 1} r_1 \sum_{p_1^{r_1}\le \frac{x}{4}} (\log p_1)^3 \sum_{r_2\ge 2} \sum_{p_2\le \left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}} (\log p_2)^2$$

$$+ \sum_{r_1\ge 1} \sum_{p_1^{r_1}\le \frac{x}{4}} (\log p_1)^2 \sum_{r_2\ge 2} r_2 \sum_{p_2\le \left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}} (\log p_2)^3$$

$$\ll \sum_{r_1\ge 1} r_1 \sum_{p_1^{r_1}\le \frac{x}{4}} (\log p_1)^3 \sum_{r_2\ge 2} \left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}} \log\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}$$

$$+ \sum_{r_1\ge 1} \sum_{p_1^{r_1}\le \frac{x}{4}} (\log p_1)^2 \sum_{r_2\ge 2} r_2 \left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}} \left(\log\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}\right)^2$$

$$\ll x^{\frac{1}{2}}(\log x) \sum_{r_1\ge 1} r_1 \sum_{p_1^{r_1}\le \frac{x}{4}} \frac{(\log p_1)^3}{p_1^{\frac{r_1}{2}}}$$

$$+ x^{\frac{1}{2}}(\log x)^2 \sum_{r_1\ge 1} r_1 \sum_{p_1^{r_1}\le \frac{x}{4}} \frac{(\log p_1)^2}{p_1^{\frac{r_1}{2}}} \ll x(\log x)^3.$$

Hence we obtain

$$S_{0,1,1,1}(x) = \frac{1}{6}x(\log x)^4 + O(x(\log x)^3). \tag{104}$$

Next, using (68)–(70) and (89), we consider

$$S_{0,2,1,0}(x) = \sum_{n\le x} \lambda^{(0)}(n)\Lambda^{(2)}(n) = \sum_{n\le x} \Lambda_2(n)\Lambda^{(2)}(n).$$

Only those n having at most two distinct prime factors contribute. We have

$$\sum_{p^r\le x} \Lambda_2(p^r)\Lambda^{(2)}(p^r) = \sum_{p^r\le x} (2r-1)(r-1)(\log p)^2$$

$$\le (\log x) \sum_{r\ge 2} \frac{(2r-1)(r-1)}{r} \sum_{p\le x^{\frac{1}{r}}} \log p$$

$$\ll (\log x) \sum_{r\ge 2} r x^{\frac{1}{r}} \ll x^{\frac{1}{2}} \log x,$$

and the contribution from n with exactly two distinct prime factors is

$$\begin{aligned}
\frac{1}{2}\sum_{p_1^{r_1}p_2^{r_2}\le x}\Lambda_2(p_1^{r_1}p_2^{r_2})\Lambda^{(2)}(p_1^{r_1}p_2^{r_2}) &= 2\sum_{p_1^{r_1}p_2^{r_2}\le x}(\log p_1)^2(\log p_2)^2\\
&= 2\sum_{p_1^{r_1}\le\frac{x}{2}}(\log p_1)^2\sum_{r_2\ge 1}\sum_{p_2\le\left(\frac{x}{p_1}\right)^{\frac{1}{r_2}}}(\log p_2)^2\\
&= 2\sum_{p_1^{r_1}\le\frac{x}{2}}(\log p_1)^2\sum_{r_2\ge 1}\left(\frac{x}{p_1}\right)^{\frac{1}{r_2}}\log\left(\left(\frac{x}{p_1}\right)^{\frac{1}{r_2}}\right)\\
&\quad+O\left(\left(\frac{x}{p_1}\right)^{\frac{1}{r_2}}\right)\\
&= 2\sum_{p_1^{r_1}\le\frac{x}{2}}(\log p_1)^2\log\left(\frac{x}{p_1^{r_1}}\right)\sum_{r_2\ge 1}\frac{1}{r_2}\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}\\
&\quad+O\left(x\sum_{p_1^{r_1}\le\frac{x}{2}}\frac{(\log p_1)^2}{p_1^{r_1}}\right)\\
&= 2\sum_{p_1^{r_1}\le\frac{x}{2}}(\log p_1)^2\log\left(\frac{x}{p_1^{r_1}}\right)\\
&\quad\left(\frac{x}{p_1^{r_1}}+O\left(\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{2}}\right)\right)+O(x(\log x)^2)\\
&= 2x\log x\sum_{r_1\ge 1}\sum_{p_1\le\left(\frac{x}{2}\right)^{\frac{1}{r_1}}}\frac{(\log p_1)^2}{p_1^{r_1}}\\
&\quad-2x\sum_{r_1\ge 1}\sum_{p_1\le\left(\frac{x}{2}\right)^{\frac{1}{r_1}}}\frac{(\log p_1)^3}{p_1^{r_1}}\\
&\quad+O\left(x^{\frac{1}{2}}\sum_{p_1^{r_1}\le\frac{x}{2}}\frac{(\log p_1)^2}{p_1^{\frac{r_1}{2}}}\right)+O(x(\log x)^2)\\
&= 2x(\log x)\left(\frac{(\log x)^2}{2}+O(1)\right)
\end{aligned}$$

$$- 2x\left(\frac{(\log x)^3}{3} + O(1)\right) + O(x(\log x)^2)$$

$$= \frac{1}{3}x(\log x)^3 + O(x(\log x)^2).$$

Hence we have

$$S_{0,2,1,0}(x) = \frac{1}{3}x(\log x)^3 + O(x(\log x)^2). \tag{105}$$

Now consider $S_{0,k,1,0}(x) = \sum_{n\le x} \lambda^{(0)}(n)\Lambda^{(k)}(n) = \sum_{n\le x} \Lambda_2(n)\Lambda^{(k)}(n), \quad (k \ge 3).$ Because of the presence of $\Lambda_2(n)$ only those n which have at most two distinct prime factors contribute to the sum. Only those prime powers $n = p^r$ with $r \ge k$ will contribute since $\Lambda^{(k)}(p^r) = 0$ for $r < k$. We see, by (69) and (70),

$$\sum_{\substack{p^r\le x\\ r\ge k}} \Lambda_2(p^r)\Lambda^{(k)}(p^r) = \sum_{r\ge k}(2r-1)\binom{r-1}{k-1}\sum_{p\le x^{\frac{1}{r}}}(\log p)^{k+2}$$

$$\ll \sum_{r\ge k}(2r-1)\binom{r-1}{k-1}\left(\frac{1}{r}\right)^{k+1}(\log x)^{k+1}\sum_{p\le x^{\frac{1}{r}}}\log p$$

$$\ll \sum_{r\ge k}\frac{1}{k!}x^{\frac{1}{r}}(\log x)^{k+1} \ll \frac{1}{k!}x^{\frac{1}{k}}(\log x)^{k+1}.$$

In calculating the contribution of n with exactly two distinct prime factors we use

$$\Lambda^{(k)}(p_1^{r_1}p_2^{r_2}) = \begin{cases} \sum_{j=1}^{k-1}\binom{k}{j}\binom{r_1-1}{j-1}\binom{r_2-1}{k-j-1}(\log p_1)^j(\log p_2)^{k-j} & \text{for } r_1+r_2 \ge k \\ 0 & \text{for } r_1+r_2 < k, \end{cases} \tag{106}$$

and without loss of generality we take $r_1 \ge \frac{k_2}{2}$. We have

$$\sum_{p_1^{r_1}p_2^{r_2}\le x}\Lambda_2(p_1^{r_1}p_2^{r_2})\Lambda^{(k)}(p_1^{r_1}p_2^{r_2})$$

$$\ll \sum_{j=1}^{k-1}\binom{k}{j}\sum_{\substack{r_1\ge\max(j,\frac{k}{2})\\ r_2\ge k-j}}\binom{r_1-1}{j-1}\binom{r_2-1}{k-j-1}\sum_{p_1^{r_1}p_2^{r_2}\le x}(\log p_1)^{j+1}(\log p_2)^{k-j+1}$$

$$\ll \sum_{j=1}^{k-1}\binom{k}{j}\sum_{\substack{r_1\geq \max(j,\frac{k}{2})\\ r_2\geq k-j}}\frac{r_1^{j-1}}{(j-1)!}\frac{r_2^{k-j-1}}{(k-j-1)!}\sum_{p_1^{r_1}\leq \frac{x}{2}}(\log p_1)^{j+1}\sum_{p_2\leq\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}}(\log p_2)^{k-j+1}$$

$$\ll \sum_{j=1}^{k-1}\binom{k}{j}\sum_{\substack{r_1\geq \max(j,\frac{k}{2})\\ r_2\geq k-j}}\frac{r_1^{j-1}}{(j-1)!}\frac{r_2^{k-j-1}}{(k-j-1)!}\sum_{p_1^{r_1}\leq \frac{x}{2}}(\log p_1)^{j+1}\left(\log\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}\right)^{k-j}\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}$$

$$\ll \sum_{j=1}^{k-1}\binom{k}{j}\sum_{r_1\geq \max(j,\frac{k}{2})}\frac{r_1^{j-1}}{(j-1)!}\sum_{p_1^{r_1}\leq \frac{x}{2}}(\log p_1)^{j+1}\left(\log\left(\frac{x}{p_1^{r_1}}\right)\right)^{k-j}\sum_{r_2\geq k-j}\frac{1}{r_2(k-j-1)!}\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}$$

$$\ll \sum_{j=1}^{k-1}\binom{k}{j}\frac{1}{(k-j)!}\sum_{r_1\geq \max(j,\frac{k}{2})}\frac{r_1^{j-1}}{(j-1)!}\sum_{p_1^{r_1}\leq \frac{x}{2}}(\log p_1)^{j+1}\left(\log\left(\frac{x}{p_1^{r_1}}\right)\right)^{k-j}\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{k-j}}$$

$$\ll (\log x)^k\sum_{j=1}^{k-1}\binom{k}{j}\frac{1}{(k-j)!j!}\sum_{r_1\geq \max(j,\frac{k}{2})}\frac{j}{r_1}\sum_{p_1\leq\left(\frac{x}{2}\right)^{\frac{1}{r_1}}}(\log p_1)\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{k-j}}$$

$$\ll \frac{(\log x)^k}{k!}\sum_{r_1\geq\frac{k}{2}}\sum_{p_1\leq\left(\frac{x}{2}\right)^{\frac{1}{r_1}}}(\log p_1)\sum_{j\leq\min(r_1,k-1)}\binom{k}{j}^2\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{k-j}}$$

$$\ll \max_{1\leq j\leq k-1}\binom{k}{j}^2\frac{(\log x)^k}{k!}\sum_{r_1\geq\frac{k}{2}}\sum_{p_1\leq\left(\frac{x}{2}\right)^{\frac{1}{r_1}}}(\log p_1)\frac{x}{p_1^{r_1}}\ll\frac{C_{19}^k x(\log x)^k}{k!},$$

since the sum over r_1 starts from $r_1\geq\lfloor\frac{k}{2}\rfloor\geq 2$. Combining these, we obtain

$$S_{0,k_2,1,0}(x)=O\left(\frac{C_{20}^{k_2}x(\log x)^{k_2}}{k_2!}\right),\qquad (k_2\geq 3). \tag{107}$$

By (66), this also settles

$$S_{0,k_2,1,1}(x)=\sum_{n\leq x}\lambda^{(0)}(n)\lambda^{(k)}(n)=\sum_{n\leq x}\Lambda_2(n)\lambda^{(k)}(n)\ll(\log x)^2\sum_{n\leq x}\Lambda_2(n)\Lambda^{(k)}(n)$$

$$=O\left(\frac{C_{21}^{k_2}x(\log x)^{k_2+2}}{k_2!}\right),\qquad (k_2\geq 3), \tag{108}$$

and there remains the case

$$S_{0,2,1,1}(x)=\sum_{n\leq x}\lambda^{(0)}(n)\lambda^{(2)}(n)=\sum_{n\leq x}\Lambda_2(n)\lambda^{(2)}(n).$$

Only those n with at most two distinct prime factors contribute to the sum. By (69) and (71), the prime powers contribute

$$\sum_{p^r\le x}\Lambda_2(p^r)\lambda^{(2)}(p^r)=\sum_{p^r\le x}(2r-1)\left(2\binom{r}{3}-\binom{r-1}{2}\right)(\log p)^6\ll\sum_{r\ge3}r^4\sum_{p\le x^{\frac{1}{r}}}(\log p)^6$$

$$\ll\sum_{r\ge3}r^4\left(\log\left(x^{\frac{1}{r}}\right)\right)^5\sum_{p\le x^{\frac{1}{r}}}\log p\ll(\log x)^5\sum_{r\ge3}\frac{x^{\frac{1}{r}}}{r}\ll x^{\frac{1}{3}}(\log x)^5.$$

To estimate the contribution from those $n=p_1^{r_1}p_2^{r_2}$ $p_1\neq p_2$, first note

$$\lambda^{(2)}(p_1^{r_1}p_2^{r_2})=(r_1-1)(r_1-2)(\log p_1)^3(\log p_2)+(r_2-1)(r_2-2)(\log p_1)(\log p_2)^3$$
$$+(8r_1r_2-3r_1-3r_2-2)(\log p_1)^2(\log p_2)^2,\tag{109}$$

as can be found from (67)–(71), so that

$$\sum_{p_1^{r_1}p_2^{r_2}\le x}\Lambda_2(p_1^{r_1}p_2^{r_2})\lambda^{(2)}(p_1^{r_1}p_2^{r_2})$$
$$\ll\sum_{p_1^{r_1}p_2^{r_2}\le x}(r_1-1)(r_1-2)(\log p_1)^4(\log p_2)^2+(r_2-1)(r_2-2)(\log p_1)^2(\log p_2)^4$$
$$+(8r_1r_2-3r_1-3r_2-2)(\log p_1)^3(\log p_2)^3.$$

The first term here contributes

$$\ll\sum_{r_1\ge2}r_1^2\sum_{p_1^{r_1}\le\frac{x}{2}}(\log p_1)^4\sum_{r_2\ge1}\sum_{p_2\le\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}}(\log p_2)^2$$

$$\ll\sum_{r_1\ge2}r_1^2\sum_{p_1^{r_1}\le\frac{x}{2}}(\log p_1)^4\sum_{r_2\ge1}\log\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}\sum_{p_2\le\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}}\log p_2$$

$$\ll(\log x)\sum_{r_1\ge2}r_1^2\sum_{p_1^{r_1}\le\frac{x}{2}}(\log p_1)^4\sum_{r_2\ge1}\frac{\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}}{r_2}\ll x(\log x)\sum_{r_1\ge2}r_1^2\sum_{p_1\le\left(\frac{x}{2}\right)^{\frac{1}{r_1}}}\frac{(\log p_1)^4}{p_1^{r_1}}$$

$$\ll x(\log x)\sum_{r_1\ge2}r_1^2\left(\log\left(\frac{x}{2}\right)^{\frac{1}{r_1}}\right)^3\sum_{p_1\le\left(\frac{x}{2}\right)^{\frac{1}{r_1}}}\frac{\log p_1}{p_1^{r_1}}\ll x(\log x)^4\sum_{r_1\ge2}\frac{1}{r_1}\sum_{p_1}\frac{\log p_1}{p_1^{r_1}}$$

$$\ll x(\log x)^4.$$

The treatment of the second term is the same. The third term contributes (since for non-zero contribution we need $r_1 + r_2 \geq 3$, we can take $r_1 \geq 2, r_2 \geq 1$)

$$\ll \sum_{r_1\geq 2}\sum_{p_1^{r_1}\leq \frac{x}{2}} (\log p_1)^3 \sum_{r_2\geq 1}(8r_1r_2 - 3r_1 - 3r_2 - 2) \sum_{p_2\leq \left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}} (\log p_2)^3$$

$$\ll (\log x)^2 \sum_{r_1\geq 2}\sum_{p_1^{r_1}\leq \frac{x}{2}} (\log p_1)^3 \sum_{r_2\geq 1} \frac{8r_1r_2 - 3r_1 - 3r_2 - 2}{r_2^2} \sum_{p_2\leq \left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}} \log p_2$$

$$\ll (\log x)^2 \sum_{r_1\geq 2}\sum_{p_1^{r_1}\leq \frac{x}{2}} (\log p_1)^3 \sum_{r_2\geq 1} \frac{8r_1r_2 - 3r_1 - 3r_2 - 2}{r_2^2} \left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}$$

$$\ll x(\log x)^2 \sum_{r_1\geq 2} r_1 \sum_{p_1\leq \left(\frac{x}{2}\right)^{\frac{1}{r_1}}} \frac{(\log p_1)^3}{p_1^{r_1}}$$

$$\ll x(\log x)^2 \sum_{r_1\geq 2} r_1 \left(\log\left(\frac{x}{2}\right)^{\frac{1}{r_1}}\right)^2 \sum_{p_1\leq \left(\frac{x}{2}\right)^{\frac{1}{r_1}}} \frac{\log p_1}{p_1^{r_1}} \ll x(\log x)^4 \sum_{r_1\geq 2}\frac{1}{r_1}\sum_{p_1}\frac{\log p_1}{p_1^{r_1}}$$

$$\ll x(\log x)^4.$$

Hence we have

$$S_{0,2,1,1}(x) \ll x(\log x)^4. \tag{110}$$

All of the formulas (104), (105), (107), (108) and (110) conform to the claim of Proposition 2, so the treatment of the cases $k_1 + k_2 + \iota_1 + \iota_2 \geq 3$ with one of k_1 and k_2 being 0 is completed.

From now on we take $k_1 + k_2 + \iota_1 + \iota_2 \geq 3$ with $1 \leq k_1 \leq k_2$. Since

$$\Delta(k_2, \iota_2; n) = \sum_{p^r|n}(\log p)\Delta\left(k_2 - 1, \iota_2; \frac{n}{p^r}\right),$$

where p denotes a prime, writing $n = ep^r$, we see

$$\begin{aligned} S_{k_1,k_2,\iota_1,\iota_2}(x) &= \sum_{p^r\leq x}(\log p)\sum_{e\leq \frac{x}{p^r}} \Delta(k_1, \iota_1; ep^r)\Delta(k_2 - 1, \iota_2; e) \\ &= \sum_{p\leq x}(\log p)\sum_{e\leq \frac{x}{p}} \Delta(k_1, \iota_1; ep)\Delta(k_2 - 1, \iota_2; e) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{\substack{p^r\le x\\ r\ge 2}}(\log p)\sum_{e\le \frac{x}{p^r}}\Delta(k_1,\iota_1;ep^r)\Delta(k_2-1,\iota_2;e)\\
&=\sum_{p\le x}(\log p)\sum_{e\le \frac{x}{p}}\Delta(k_1,\iota_1;ep)\Delta(k_2-1,\iota_2;e)+\Sigma_8,\ \text{say},\\
&=\sum_{p\le x}(\log p)\sum_{\substack{e\le \frac{x}{p}\\ (e,p)=1}}\Delta(k_1,\iota_1;ep)\Delta(k_2-1,\iota_2;e)\\
&\quad+\sum_{p\le x}(\log p)\sum_{\substack{e\le \frac{x}{p}\\ (e,p)=p}}\Delta(k_1,\iota_1;ep)\Delta(k_2-1,\iota_2;e)+\Sigma_8\\
&=\sum_{p\le x}(\log p)\sum_{\substack{e\le \frac{x}{p}\\ (e,p)=1}}\Delta(k_1,\iota_1;ep)\Delta(k_2-1,\iota_2;e)+\Sigma_7+\Sigma_8,\ \text{say},
\end{aligned}
\tag{111}
$$

Now we use Lemma 3 for $\Delta(k_1,\iota_1;ep)$, and have

$$
\begin{aligned}
&\sum_{p\le x}(\log p)\sum_{\substack{e\le \frac{x}{p}\\ (e,p)=1}}\Delta(k_1,\iota_1;ep)\Delta(k_2-1,\iota_2;e)\\
&=k_1\sum_{p\le x}(\log p)^2\sum_{\substack{e\le \frac{x}{p}\\ (e,p)=1}}\Delta(k_1-1,\iota_1;ep)\Delta(k_2-1,\iota_2;e)\\
&\quad+2[\iota_1=1]\sum_{p\le x}(\log p)^2\sum_{\substack{e\le \frac{x}{p}\\ (e,p)=1}}\Delta(k_1+1,0;e)\Delta(k_2-1,\iota_2;e)\\
&\quad+[\iota_1=1]\sum_{p\le x}(\log p)^3\sum_{\substack{e\le \frac{x}{p}\\ (e,p)=1}}\Delta(k_1,0;e)\Delta(k_2-1,\iota_2;e)\\
&=k_1\sum_{p\le x}(\log p)^2\sum_{e\le \frac{x}{p}}\Delta(k_1-1,\iota_1;e)\Delta(k_2-1,\iota_2;e)\\
&\quad+2[\iota_1=1]\sum_{p\le x}(\log p)^2\sum_{e\le \frac{x}{p}}\Delta(k_1+1,0;e)\Delta(k_2-1,\iota_2;e)\\
&\quad+[\iota_1=1]\sum_{p\le x}(\log p)^3\sum_{e\le \frac{x}{p}}\Delta(k_1,0;e)\Delta(k_2-1,\iota_2;e)
\end{aligned}
$$

$$
\begin{aligned}
&- k_1 \sum_{p\le x} (\log p)^2 \sum_{\substack{e\le \frac{x}{p}\\ (e,p)=p}} \Delta(k_1-1,\iota_1;e)\Delta(k_2-1,\iota_2;e)\\
&- 2[\iota_1=1]\sum_{p\le x} (\log p)^2 \sum_{\substack{e\le \frac{x}{p}\\ (e,p)=p}} \Delta(k_1+1,0;e)\Delta(k_2-1,\iota_2;e)\\
&- [\iota_1=1]\sum_{p\le x} (\log p)^3 \sum_{\substack{e\le \frac{x}{p}\\ (e,p)=p}} \Delta(k_1,0;e)\Delta(k_2-1,\iota_2;e)\\
&= \Sigma_1+\Sigma_2+\Sigma_3-\Sigma_4-\Sigma_5-\Sigma_6, \text{ say,} \qquad (112)
\end{aligned}
$$

so that

$$
S_{k_1,k_2,\iota_1,\iota_2}(x)=\Sigma_1+\Sigma_2+\Sigma_3-\Sigma_4-\Sigma_5-\Sigma_6+\Sigma_7+\Sigma_8. \qquad (113)
$$

Applying the induction hypothesis to Σ_1, Σ_2 and Σ_3 we have

$$
\begin{aligned}
\Sigma_1 &= k_1 \sum_{p\le x} (\log p)^2 \sum_{e\le \frac{x}{p}} \Delta(k_1-1,\iota_1;e)\Delta(k_2-1,\iota_2;e)\\
&= x\frac{k_1 P(k_1-1,k_2-1,\iota_1,\iota_2)}{(k_1+k_2+2\iota_1+2\iota_2-3)!}\sum_{p\le x}\frac{(\log p)^2}{p}\left(\log\frac{x}{p}\right)^{k_1+k_2+2\iota_1+2\iota_2-3}\\
&\quad + O\left(\frac{xC_{18}^{k_1+k_2-2}}{\max(k_1-1,k_2-1)!}\sum_{p\le x}\frac{(\log p)^2}{p}\left(\log\frac{x}{p}\right)^{k_1+k_2+2\iota_1+2\iota_2-4}\right),\\
\Sigma_2 &= 2[\iota_1=1]\sum_{p\le x}(\log p)^2 \sum_{\substack{e\le \frac{x}{p}\\ (e,p)=1}} \Delta(k_1+1,0;e)\Delta(k_2-1,\iota_2;e)\\
&= 2[\iota_1=1]x\frac{P(k_1+1,k_2-1,0,\iota_2)}{(k_1+k_2+2\iota_1+2\iota_2-3)!}\sum_{p\le x}\frac{(\log p)^2}{p}\left(\log\frac{x}{p}\right)^{k_1+k_2+2\iota_1+2\iota_2-3}\\
&\quad + O\left([\iota_1=1]\frac{xC_{18}^{k_1+k_2}}{\max(k_1+1,k_2-1)!}\sum_{p\le x}\frac{(\log p)^2}{p}\left(\log\frac{x}{p}\right)^{k_1+k_2+2\iota_1+2\iota_2-4}\right),\\
\Sigma_3 &= [\iota_1=1]\sum_{p\le x}(\log p)^3\sum_{e\le \frac{x}{p}}\Delta(k_1,0;e)\Delta(k_2-1,\iota_2;e)\\
&= [\iota_1=1]x\frac{P(k_1,k_2-1,0,\iota_2)}{(k_1+k_2+2\iota_1+2\iota_2-4)!}\sum_{p\le x}\frac{(\log p)^3}{p}\left(\log\frac{x}{p}\right)^{k_1+k_2+2\iota_1+2\iota_2-4}
\end{aligned}
$$

$$+ O\left([\iota_1 = 1]\frac{xC_{18}^{k_1+k_2-1}}{\max(k_1,k_2-1)!}\sum_{p\le x}\frac{(\log p)^3}{p}\left(\log\frac{x}{p}\right)^{k_1+k_2+2\iota_1+2\iota_2-5}\right). \tag{114}$$

We must mention that some care is needed in ensuring the applicability of the induction hypothesis. In these applications we need to check whether the hypothesis of Proposition 2 is satisfied. Specifically, for Σ_1 we need to have $k_1 - 1 + \iota_1 \ge 1$, $k_2 - 1 + \iota_2 \ge 1$. If these are not satisfied, then either $k_2 + \iota_2 = 1$ or $k_1 + \iota_1 = 1$. Recall that we are working under the condition $k_1+k_2+\iota_1+\iota_2 \ge 3$ with $1 \le k_1 \le k_2$. If $k_2 + \iota_2 = 1$, then $k_2 = 1, \iota_2 = 0$, and therefore $k_1 = 1$, $\iota_1 = 1$. So we encounter

$$\begin{aligned} S_{1,1,1,0}(x) &= \sum_{n\le x}\lambda^{(1)}(n)\Lambda(n) = \sum_{r\ge 2}(r-1)^2\sum_{p\le x^{\frac{1}{r}}}(\log p)^4 \\ &\le \sum_{r\ge 2}(r-1)^2\Big(\log(x^{\frac{1}{r}})\Big)^3\sum_{p\le x^{\frac{1}{r}}}\log p \ll (\log x)^3\sum_{r\ge 2}\frac{1}{r} \ll x^{\frac{1}{2}}(\log x)^3, \end{aligned} \tag{115}$$

which is compatible with what (99)–(100) written for $(k_1, k_2, \iota_1, \iota_2) = (1, 1, 1, 0)$ gives, namely

$$S_{1,1,1,0}(x) = 0 + O\big(C_{18}^2 x(\log x)^2\big).$$

If $k_1 + \iota_1 = 1$, then $k_1 = 1, \iota_1 = 0$ and $k_2 + \iota_2 \ge 2$. If $k_2 = 1$, then $\iota_2 = 1$ too, and we encounter $S_{1,1,0,1}(x)$ which is the same as $S_{1,1,1,0}(x)$ just settled in (115). If $k_2 \ge 2$, then $\iota_2 = 0$ or $\iota_2 = 1$, and we encounter $S_{1,k_2,0,0}(x)$ or $S_{1,k_2,0,1}(x)$. Now, with $k \ge 2$, we have

$$\begin{aligned} S_{1,k,0,0}(x) &= \sum_{n\le x}\Lambda(n)\Lambda^{(k)}(n) = \sum_{r\ge k}\binom{r-1}{k-1}\sum_{p\le x^{\frac{1}{r}}}(\log p)^{k+1} \\ &\le \sum_{r\ge k}\binom{r-1}{k-1}\Big(\log(x^{\frac{1}{r}})\Big)^k\sum_{p\le x^{\frac{1}{r}}}(\log p) \ll \frac{(\log x)^k}{(k-1)!}\sum_{r\ge k}\frac{1}{r}x^{\frac{1}{r}} \ll \frac{x^{\frac{1}{k}}(\log x)^k}{k!}, \end{aligned} \tag{116}$$

and this is compatible with

$$S_{1,k,0,0}(x) = 0 + O\left(\frac{C_{18}^{k+1}(\log x)^{k-1}}{k!}\right), \quad (k \ge 2)$$

which is (99)–(100) written for $(k_1, k_2, \iota_1, \iota_2) = (1, k, 0, 0)$. Similarly, for $k \ge 2$,

$$S_{1,k,0,1}(x) = \sum_{n\le x} \Lambda(n)\lambda^{(k)}(n) = \sum_{r\ge k+1} \binom{r-1}{k} \frac{2r-k-1}{k+1} \sum_{p\le x^{\frac{1}{r}}} (\log p)^{k+3}$$

$$\le \sum_{r\ge k+1} \binom{r-1}{k} \frac{2r-k-1}{k+1} \Big(\log(x^{\frac{1}{r}})\Big)^{k+2} \sum_{p\le x^{\frac{1}{r}}} (\log p)$$

$$\ll \frac{(\log x)^{k+2}}{(k+1)!} \sum_{r\ge k+1} \frac{1}{r} x^{\frac{1}{r}} \ll \frac{x^{\frac{1}{k+1}} (\log x)^{k+2}}{(k+2)!}, \tag{117}$$

which is compatible with

$$S_{1,k,0,1}(x) = 0 + O\left(\frac{C_{18}^{k+1}(\log x)^{k+1}}{k!}\right), \quad (k \ge 2)$$

obtained by writing (99)–(100) for $(k_1, k_2, \iota_1, \iota_2) = (1, k, 0, 1)$. An examination of Σ_2 and Σ_3 shows that the only new situation to check is for Σ_2 with $k_2 = 1, \iota_2 = 0$, in which case $k_1 = 1$, and the sum to be checked is $S_{2,0,0,0}(x)$. But, trivially $S_{2,0,0,0}(x) = 0$, and this is compatible with $S_{2,0,0,0}(x) = 0 + O(C_{18}^2 x)$ which is (99)–(100) written with $(k_1, k_2, \iota_1, \iota_2) = (2, 0, 0, 0)$. Hence the expressions in (114) are completely justified, and in the few cases where the exponents of $\log \frac{x}{p}$ in the sums in the error terms may be negative the statement of Proposition 2 has been verified independently.

We evaluate the inner sums (over p) in (114) using a variant of (87),

$$\sum_{p\le x} \frac{(\log p)^k}{p} \left(\log \frac{x}{p}\right)^{\ell} = \frac{(k-1)!\ell!}{(k+\ell)!} (\log x)^{k+\ell} + O\left(\frac{(k-1)!\ell!}{(k+\ell-1)!} (\log x)^{k+\ell-1}\right) \tag{118}$$

for $k \ge 1, \ell \ge 0$, which is Lemma 5.4 of [5], and for $\Sigma_1 + \Sigma_2 + \Sigma_3$ we use

$$k_1 P(k_1 - 1, k_2 - 1, \iota_1, \iota_2) + 2[\iota_1 = 1]\big(P(k_1 + 1, k_2 - 1, 0, \iota_2) + P(k_1, k_2 - 1, 0, \iota_2)\big)$$
$$= P(k_1, k_2, \iota_1, \iota_2) \tag{119}$$

which can be verified from the definition of $P(k_1, k_2, \iota_1, \iota_2)$. So we obtain

$$\Sigma_1 + \Sigma_2 + \Sigma_3$$
$$= \frac{P(k_1, k_2, \iota_1, \iota_2)}{(k_1 + k_2 + 2\iota_1 + 2\iota_2 - 1)!} x (\log x)^{k_1+k_2+2\iota_1+2\iota_2-1} \cdot \left(1 + O\left(\frac{k_2}{\log x}\right)\right)$$

$$+ O\left(\frac{C_{18}^{k_1+k_2} x(\log x)^{k_1+k_2+2\iota_1+2\iota_2-2}}{(k_2+1)!} \cdot \left(1+\frac{k_2}{\log x}\right)\right)$$
$$= \frac{P(k_1,k_2,\iota_1,\iota_2)}{(k_1+k_2+2\iota_1+2\iota_2-1)!} x(\log x)^{k_1+k_2+2\iota_1+2\iota_2-1}$$
$$+ O\left(\frac{C_{18}^{k_1+k_2} x(\log x)^{k_1+k_2+2\iota_1+2\iota_2-2}}{(\max\{k_1,k_2\})!}\right). \tag{120}$$

Here, in writing the middle member of the formula, we have invoked $k_2 \geq k_1$ which was taken as a convention, and in passing to the last line we have used

$$\frac{k_2 P(k_1,k_2,\iota_1,\iota_2)}{(k_1+k_2+2\iota_1+2\iota_2-1)!} \ll \frac{k_2+1)!}{(2k_2-1)!},$$

in case the left-hand side is not 0, which is seen from (100).

It remains to show that $\Sigma_4, \ldots, \Sigma_8$ which were defined in (111) and (112) can all be included in the error term of (99). In Σ_4 letting $e = e'p$, and then using Lemmas 3 and 4, we have

$$\Sigma_4 = k_1 \sum_{p\leq x} (\log p)^2 \sum_{\substack{e\leq \frac{x}{p} \\ (e,p)=p}} \Delta(k_1-1,\iota_1;e)\Delta(k_2-1,\iota_2;e)$$
$$\ll k_1^2 (\log x)^{k_1+2\iota_1-2} \sum_{p\leq x} (\log p)^3 \sum_{e'\leq \frac{x}{p^2}} \Delta(k_2-1,\iota_2;pe')$$
$$\ll \frac{C_2^{k_2-1} k_1^2}{(k_2+2\iota_2-2)!} x(\log x)^{k_1+k_2+2\iota_1+2\iota_2-4} \sum_{p\leq x} \frac{(\log p)^3}{p^2}$$
$$\ll \frac{C_{18}^{k_1+k_2} x(\log x)^{k_1+k_2+2\iota_1+2\iota_2-2}}{k_2!} \tag{121}$$

provided that C_{18} is chosen sufficiently large. It is easily seen that $\Sigma_5 - \Sigma_8$ are handled similarly. This completes the proof of Proposition 2. □

Applying summation by parts to the result of Proposition 2, we immediately obtain

Proposition 3

$$\sum_{n\leq x} \Delta(k_1,\iota_1;n)\Delta(k_2,\iota_2;n)n = \frac{P(k_1,k_2,\iota_1,\iota_2)x^2(\log x)^{k_1+k_2+2\iota_1+2\iota_2-1}}{2(k_1+k_2+2\iota_1+2\iota_2-1)!}$$

$$+ O\left(\frac{C_{18}^{k_1+k_2}x^2(\log x)^{k_1+k_2+2\iota_1+2\iota_2-2}}{(\max\{k_1,k_2\})!}\right); \tag{122}$$

$$\sum_{n>x}\frac{\Delta(k_1,\iota_1;n)\Delta(\ell,\iota_2;n)}{n^3} = \frac{P(k_1,k_2,\iota_1,\iota_2)(\log x)^{k_1+k_2+2\iota_1+2\iota_2-1}}{2x^2(k_1+k_2+2\iota_1+2\iota_2-1)!}$$
$$+ O\left(\frac{C_{18}^{k_1+k_2}(\log x)^{k_1+k_2+2\iota_1+2\iota_2-2}}{x^2(\max\{k_1,k_2\})!}\right). \tag{123}$$

7 Correlation of Zeta Zeros with the Relative Maxima of $|\zeta(\frac{1}{2}+it)|$

Assuming RH, let $\varrho = \frac{1}{2}+i\upsilon,\ \upsilon > 0$, run through the non-trivial zeros of $Z_1(s)$. For correlations of zeta zeros with the relative maxima of the Riemann zeta-function on the critical line we set out from (22) by writing

$$\sum_{\gamma}\sum_{\frac{T}{2}<\upsilon\le T}\frac{4x^{i(\gamma-\upsilon)}}{4+(\gamma-\upsilon)^2} \tag{124}$$
$$= -x^{-2}\sum_{n\le x}\Lambda(n)n^2\sum_{\frac{T}{2}<\upsilon\le T}n^{-\varrho} - x^2\sum_{n>x}\Lambda(n)n^{-2}\sum_{\frac{T}{2}<\upsilon\le T}n^{-\varrho}$$
$$+ \frac{T\log^2 T}{4\pi x^2}\left(1+O\left(\frac{1}{\log T}\right)\right) + O\left(\frac{x^{\frac{1}{2}}\log T}{T}\right) + O(x^{-\frac{5}{2}}\log T)$$

for $x \ge 1$ and $T \ge T_0$. We see that we need to calculate

$$\sum_{\frac{T}{2}<\upsilon\le T}n^{-\varrho} \quad \text{for } n = p^a,\ p:\text{prime},\ a\in\mathbb{Z}^+. \tag{125}$$

Just as in the pair correlation of zeta zeros we do not hope to get an asymptotic result for $x > T$, so that if we assume $x \le T^M$, then for the tail of the sum over $n > x$ in (124) we use

$$x^2\sum_{n>T^M}\Lambda(n)n^{-2}\sum_{\frac{T}{2}<\upsilon\le T}n^{-\varrho} \ll x^2T\log T\sum_{n>T^M}\frac{\Lambda(n)}{n^{\frac{5}{2}}} \ll x^2T^{1-\frac{3M}{2}}\log T. \tag{126}$$

We will see towards the end of this calculation that any fixed $M > \frac{4}{3}$ can be employed. Upon this simplification we write

$$\sum_{\gamma}\sum_{\frac{T}{2}<\upsilon\le T}\frac{4x^{i(\gamma-\upsilon)}}{4+(\gamma-\upsilon)^2} \tag{127}$$
$$= -x^{-2}\sum_{n\le x}\Lambda(n)n^2\sum_{\frac{T}{2}<\upsilon\le T}n^{-\varrho} - x^2\sum_{x<n\le T^M}\Lambda(n)n^{-2}\sum_{\frac{T}{2}<\upsilon\le T}n^{-\varrho}$$
$$+\frac{T\log^2 T}{4\pi x^2}\left(1+O\left(\frac{1}{\log T}\right)\right)+O\left(\frac{x^{\frac{1}{2}}\log T}{T}\right)+O\left(\frac{\log T}{x^{\frac{5}{2}}}\right)$$
$$+O\left(x^2T^{1-\frac{3M}{2}}\log T\right)$$

for $1\le x<T^M$ and $T\ge T_0$.

We are now going to calculate, more generally than (125), the expression

$$\sum_{\frac{T}{2}<\upsilon\le T}n^{-\varrho}\left(\frac{\chi'}{\chi}(\varrho+b)\right)^{-k}\quad\text{for }\ 2\le n\le T^M,\ 0\le k\le\left\lfloor\frac{\log T}{\log\log T}\right\rfloor,\ b=-2\text{ or }2 \tag{128}$$

(where n and k are integers) for later use in Sect. 8; the special case of n a prime power and $k=0$ is (125). The result will be formula (149) in Proposition 4 below.

To calculate (128) we will use

$$\sum_{\frac{T}{2}<\upsilon\le T}n^{-\varrho}\left(\frac{\chi'}{\chi}(\varrho+b)\right)^{-k}=\frac{1}{2\pi i}\oint_{\mathscr{C}}\frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(s+b)\right)^{-k}n^{-s}\,ds$$
$$+O\left(n^{-\frac{1}{2}}(\log T)^{1-k}\right), \tag{129}$$

where the contour $\mathscr{C}$ is the counterclockwise traversed rectangle having vertices at $-\delta+it_r, 1+\delta+it_r,\ (r=1,2)$, with

$$\delta=\frac{1}{\log T},\ t_r=\frac{rT}{2}+O(1),\ (r=1,2), \tag{130}$$

chosen such that the horizontal sides of $\mathscr{C}$ are a distance $\gg\frac{1}{\log T}$ away from the zeros of $Z_1(s)$. The error term in (129) is the effect of such a choice estimated trivially. Just as the estimate for $\frac{\zeta'}{\zeta}(s)$ (see Chap. 17 of [2]), from an argument based upon the representation of $\frac{Z_1'}{Z_1}(s)$ as a sum of partial fractions (which is similar to that for $\frac{\zeta'}{\zeta}(s)$ since the count of zeros is almost the same) we see

$$\frac{Z_1'}{Z_1}(\sigma+it_r)=O(\log^2 T),\quad(-1\le\sigma\le 2), \tag{131}$$

so that the integrals along the horizontal sides of $\mathscr{C}$ satisfy

$$\int_{-\delta\pm t_r}^{1+\delta\pm t_r} \frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(s+b)\right)^{-k} n^{-s}\,ds \ll \frac{n^{\delta}(\log T)^{2-k}}{\log n} \ll \frac{(\log T)^{2-k}}{\log n}. \tag{132}$$

The integral along the right vertical side of $\mathscr{C}$ becomes, upon using (64) and (16),

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{1+\delta+it_1}^{1+\delta+it_2} \frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(s+b)\right)^{-k} n^{-s}\,ds \qquad (133)\\
&= -\frac{(-1)^k}{2\pi}\sum_{\ell=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor} 2^{\ell}\sum_{m=2}^{\infty}\frac{\Lambda^{(\ell+1)}(m)}{(mn)^{1+\delta}}\int_{t_1}^{t_2}\frac{(mn)^{-it}}{\left(\log\frac{t}{2\pi}\right)^{k+\ell}}\,dt\\
&\quad+\frac{(-1)^k}{2\pi}\sum_{\ell=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor} 2^{\ell+1}\sum_{m=2}^{\infty}\frac{\lambda^{(\ell)}(m)}{(mn)^{1+\delta}}\int_{t_1}^{t_2}\frac{(mn)^{-it}}{\left(\log\frac{t}{2\pi}\right)^{k+\ell+1}}\,dt\\
&\quad+O\left(\frac{k}{n}\int_{t_1}^{t_2}\frac{\left|\frac{Z_1'}{Z_1}(1+\delta+it)\right|}{t\left(\log\frac{t}{2\pi}\right)^{k+1}}\right)dt\\
&\quad+O\left(\frac{(\log T)^{-k}}{n}\exp\left(\frac{C_{22}\log T\log\log\log T}{\log\log T}\right)\right).
\end{aligned}$$

By (58)–(60),

$$\left|\frac{Z_1'}{Z_1}(1+\delta+it)\right| \ll \log\log|t|, \quad (|t|\ge t_0), \tag{134}$$

so the first error term in (133) is trivially $\ll \dfrac{k\log\log T}{n(\log T)^{k+1}} \ll \dfrac{(\log T)^{-k}}{n}$. The integrals here are handled by the following lemma.

Lemma 5 *For A large,*

$$\int_A^B w^{it}\frac{dt}{\left(\log\frac{t}{2\pi}\right)^k} = \begin{cases} O\left(\frac{1}{|\log w|(\log A)^k}\right) & \text{if } w\neq 1,\\ \frac{B-A}{\left(\log\frac{A}{2\pi}\right)^k}+O\left(\frac{kA}{(\log A)^{k+1}}\right) & \text{if } w=1,\end{cases} \tag{135}$$

where $A<B\le 2A$, $w>0$, $k\in\mathbb{N}$, $k=o(\log A)$. The constants implied in the O-terms are absolute.

Proof We recall Lemma 4.3 of [21]: Let $F(x)$, $G(x)$ be real functions, $G(x)/F'(x)$ monotonic and $|F'(x)/G(x)|\ge m>0$ throughout the interval $[a,b]$. Then

$$\left|\int_A^B G(x)e^{iF(x)}dx\right| \le \frac{4}{m}.$$

Using this, if $w \neq 1$, we have

$$\left| \int_A^B w^{it} \frac{dt}{\left(\log \frac{t}{2\pi}\right)^k} \right| \leq \frac{4}{|\log w| \left(\log \frac{A}{2\pi}\right)^k}.$$

By Bernoulli's inequality, we have

$$\frac{1}{\left(\log \frac{A}{2\pi}\right)^k} = \frac{1}{(\log A)^k \left(1 - \frac{\log 2\pi}{\log A}\right)^k} \leq \frac{1}{(\log A)^k \left(1 - k\frac{\log 2\pi}{\log A}\right)} \ll \frac{1}{(\log A)^k},$$

since $k = o(\log A)$, which completes the proof of the case $w \neq 1$.

If $w = 1$, then integration by parts gives

$$\begin{aligned} \int_A^B \frac{dt}{\left(\log \frac{t}{2\pi}\right)^k} &= \frac{t}{\left(\log \frac{t}{2\pi}\right)^k}\Bigg|_A^B + k\int_A^B \frac{dt}{\left(\log \frac{t}{2\pi}\right)^{k+1}} \\ &= \frac{B}{\left(\log \frac{B}{2\pi}\right)^k} - \frac{A}{\left(\log \frac{A}{2\pi}\right)^k} + O\left(\frac{k(B-A)}{\left(\log \frac{A}{2\pi}\right)^{k+1}}\right). \end{aligned}$$

Since $A < B \leq 2A$, the last error term is $\ll \frac{kA}{(\log \frac{A}{2\pi})^{k+1}} \ll \frac{kA}{(\log A)^{k+1}}$. To finish the proof we simply observe that $\frac{1}{\left(\log \frac{B}{2\pi}\right)^k} = \frac{1}{\left(\log \frac{A}{2\pi}\right)^k} + O\left(\frac{k}{(\log A)^{k+1}}\right)$ by the mean value theorem. □

By Lemma 5, and

$$\frac{\zeta^{(j)}}{\zeta}(s) \sim \frac{(-1)^j j!}{(s-1)^j}, \quad (s \to 1;\ j \in \mathbb{Z}^+) \tag{136}$$

employed in (62), we see that

$$\begin{aligned} &\sum_{\ell=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} 2^\ell \sum_{m=2}^{\infty} \frac{\Lambda^{(\ell+1)}(m)}{(mn)^{1+\delta}} \int_{t_1}^{t_2} \frac{(mn)^{-it}}{\left(\log \frac{t}{2\pi}\right)^{k+\ell}}\, dt \\ &\quad \ll \sum_{\ell=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} 2^\ell \sum_{m=2}^{\infty} \frac{\Lambda^{(\ell+1)}(m)}{(mn)^{1+\delta}} \frac{1}{(\log mn)(\log T)^{k+\ell}} \\ &\quad \ll \frac{(\log T)^{-k}}{n^{1+\delta}\log n} \sum_{\ell=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} \left(\frac{2}{\log T}\right)^\ell \sum_{m=2}^{\infty} \frac{\Lambda^{(\ell+1)}(m)}{m^{1+\delta}} \end{aligned}$$

$$\ll \frac{(\log T)^{-k}}{n\log n}\sum_{\ell=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor}\left(\frac{2}{\log T}\right)^{\ell}\left(-\frac{\zeta'}{\zeta}(1+\delta)\right)^{\ell+1}$$

$$\ll \frac{(\log T)^{-k}}{\delta n\log n}\sum_{\ell=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor}\left(\frac{3}{\delta\log T}\right)^{\ell} \ll \frac{(\log T)^{1-k}}{n\log n}\exp\left(\frac{C_{23}\log T}{\log\log T}\right). \tag{137}$$

A similar calculation gives that the same kind of bound holds for the part involving $\lambda^{(\ell)}$ in the right-hand side of (133). The bound in (137) is smaller than the last error term of (133), so we obtain

$$\frac{1}{2\pi i}\int_{1+\delta+it_1}^{1+\delta+it_2}\frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(s+b)\right)^{-k}n^{-s}\,ds \ll \frac{(\log T)^{-k}}{n}\exp\left(\frac{C_{24}\log T\log\log\log T}{\log\log T}\right). \tag{138}$$

It remains to calculate the integral along the left vertical side of $\mathscr{C}$. In this integral we replace s by $1-s$, using

$$\frac{Z_1'}{Z_1}(s) = \frac{\chi'}{\chi}(s) - \frac{Z_1'}{Z_1}(1-s) \tag{139}$$

which follows from (56), so that we have

$$\int_{-\delta+it_1}^{-\delta+it_2}\frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(s+b)\right)^{-k}n^{-s}ds = -\int_{1+\delta-it_1}^{1+\delta-it_2}\frac{\chi'}{\chi}(s)\left(\frac{\chi'}{\chi}(1-s+b)\right)^{-k}n^{s-1}ds$$
$$+\int_{1+\delta-it_1}^{1+\delta-it_2}\frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(1-s+b)\right)^{-k}n^{s-1}ds. \tag{140}$$

Integrating by parts and then using the estimates (16) and (17) we see that

$$\int_{1+\delta-it_1}^{1+\delta-it_2}\frac{\chi'}{\chi}(s)\left(\frac{\chi'}{\chi}(1-s+b)\right)^{-k}n^{s-1}\,ds$$
$$= \frac{\chi'}{\chi}(s)\left(\frac{\chi'}{\chi}(1-s+b)\right)^{-k}\frac{n^{s-1}}{\log n}\Big|_{1+\delta-it_2}^{1+\delta-it_1}$$
$$-\frac{1}{\log n}\int_{1+\delta-it_2}^{1+\delta-it_1}\left(\frac{\chi'}{\chi}\right)'(s)\left(\frac{\chi'}{\chi}(1-s+b)\right)^{-k}n^{s-1}\,ds$$
$$-\frac{1}{\log n}\int_{1+\delta-it_2}^{1+\delta-it_1}\frac{\chi'}{\chi}(s)\frac{d}{ds}\left\{\left(\frac{\chi'}{\chi}(1-s+b)\right)^{-k}\right\}n^{s-1}ds \ll \frac{(\log T)^{1-k}}{\log n}. \tag{141}$$

Next, letting $s = 1 + \delta - it$, and using (134), (64) and (16), we have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{1+\delta-it_1}^{1+\delta-it_2}\frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(1-s+b)\right)^{-k}n^{s-1}\,ds\\
&\quad=\frac{1}{2\pi i}\int_{1+\delta-i\frac{T}{2}}^{1+\delta-iT}\frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(1-s+b)\right)^{-k}n^{s-1}\,ds+O\left(\frac{\log\log T}{(\log T)^k}\right)\\
&\quad=-\frac{(-1)^k n^\delta}{2\pi}\sum_{\ell=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}2^\ell\sum_{m=2}^{\infty}\frac{\Lambda^{(\ell+1)}(m)}{m^{1+\delta}}\int_{\frac{T}{2}}^{T}\left(\frac{m}{n}\right)^{it}\frac{dt}{\left(\log\frac{t}{2\pi}\right)^{k+\ell}}\\
&\qquad+\frac{(-1)^k n^\delta}{2\pi}\sum_{\ell=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}2^{\ell+1}\sum_{m=2}^{\infty}\frac{\lambda^{(\ell)}(m)}{m^{1+\delta}}\int_{\frac{T}{2}}^{T}\left(\frac{m}{n}\right)^{it}\frac{dt}{\left(\log\frac{t}{2\pi}\right)^{k+\ell+1}}\\
&\qquad+O\left(k\int_{\frac{T}{2}}^{T}\left|\frac{Z_1'}{Z_1}(1+\delta-it)\right|\left(\log\frac{t}{2\pi}\right)^{-k-1}\frac{dt}{t}\right)\\
&\qquad+O\left((\log T)^{-k}\exp\left(\frac{C_{25}\log T\log\log\log T}{\log\log T}\right)\right).
\end{aligned}\tag{142}$$

Putting together our findings as of (129), we can write

$$\begin{aligned}
&\sum_{\frac{T}{2}<\upsilon\le T}n^{-\varrho}\left(\frac{\chi'}{\chi}(\varrho+b)\right)^{-k}\\
&\quad=-\frac{(-1)^k n^\delta}{2\pi}\sum_{\ell=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}2^\ell\sum_{m=2}^{\infty}\frac{\Lambda^{(\ell+1)}(m)}{m^{1+\delta}}\int_{\frac{T}{2}}^{T}\left(\frac{m}{n}\right)^{it}\frac{dt}{\left(\log\frac{t}{2\pi}\right)^{k+\ell}}\\
&\qquad+\frac{(-1)^k n^\delta}{2\pi}\sum_{\ell=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}2^{\ell+1}\sum_{m=2}^{\infty}\frac{\lambda^{(\ell)}(m)}{m^{1+\delta}}\int_{\frac{T}{2}}^{T}\left(\frac{m}{n}\right)^{it}\frac{dt}{\left(\log\frac{t}{2\pi}\right)^{k+\ell+1}}\\
&\qquad+O\left((\log T)^{-k}\exp\left(\frac{C_{26}\log T\log\log\log T}{\log\log T}\right)\right).
\end{aligned}\tag{143}$$

We split the sums over m into three parts: the $m = n$ term, the terms with $0 < |m-n| < \frac{n}{2}$, the terms with $|m-n| \ge \frac{n}{2}$. The $m = n$ term gives

$$\frac{(-1)^k}{2\pi n}\int_{\frac{T}{2}}^{T}\sum_{\ell=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}2^\ell\left(\frac{2\lambda^{(\ell)}(n)}{(\log\frac{t}{2\pi})^{k+\ell+1}}-\frac{\Lambda^{(\ell+1)}(n)}{(\log\frac{t}{2\pi})^{k+\ell}}\right)dt,\tag{144}$$

and it will be seen that this is the main term. By Lemma 5, the contribution from the $m \neq n$ terms to (143) is

$$\ll \frac{n^{\delta}}{(\log T)^k} \sum_{\ell=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} \left(\frac{2}{\log T}\right)^{\ell} \sum_{m \neq n} \frac{\Lambda^{(\ell+1)}(m) + \frac{2}{\log T}\lambda^{(\ell)}(m)}{m^{1+\delta} |\log \frac{m}{n}|}. \tag{145}$$

For $m \leq \frac{n}{2}$ and $m \geq \frac{3n}{2}$, we have $|\log \frac{m}{n}| \gg 1$, so estimating similarly to (137), the contribution from the terms with $|m-n| \geq \frac{n}{2}$ is seen to be

$$\ll (\log T)^{-k} \exp\left(\frac{C_{27} \log T}{\log\log T}\right). \tag{146}$$

For the terms with $0 < |m-n| < \frac{n}{2}$ we use $|\log \frac{m}{n}| \gg \frac{|n-m|}{m}$, so the contribution of these terms to (143) is

$$\ll (\log T)^{-k} \sum_{\ell=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} \left(\frac{2}{\log T}\right)^{\ell} \sum_{0<|m-n|<\frac{n}{2}} \frac{\Lambda^{(\ell+1)}(m) + \frac{2}{\log T}\lambda^{(\ell)}(m)}{|m-n|}. \tag{147}$$

Estimating the inner sum by Lemmas 1 and 2, we see that (147) is majorized by

$$\begin{aligned}
&(\log T)^{-k} \sum_{\ell=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} \left(\frac{2\log 2n}{\log T}\right)^{\ell+1} \left\{\frac{(\log T)(\log\log 3n+7)_{\ell+1}}{\ell!} + \frac{(\log 2n)(\log\log 3n+7)_{\ell+2}}{(\ell+1)!}\right\} \\
&\ll (\log T)^{1-k} \exp\left(C_{28}\frac{\log T}{\log\log T}\right) \sum_{\ell=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} \frac{(\log\log 3n+9)_{\ell}}{\ell!} \\
&\ll (\log T)^{-k} \exp\left(C_{29}\frac{\log T}{\log\log T}\right) \frac{(\log\log 3n+10)_{\lfloor \frac{\log T}{\log\log T} \rfloor}}{\lfloor \frac{\log T}{\log\log T} \rfloor!} \\
&= (\log T)^{-k} \exp\left(C_{29}\frac{\log T}{\log\log T}\right) \exp\left(\sum_{j=\lfloor \log\log 3n \rfloor+10}^{\lfloor \frac{\log T}{\log\log T} \rfloor} \log\left(1+\frac{\log\log 3n+9}{j}\right)\right) \\
&\quad \times \prod_{1 \leq j \leq \lfloor \log\log 3n \rfloor+9} \left(1+\frac{\log\log 3n+9}{j}\right) \\
&\leq (\log T)^{-k} \exp\left(C_{29}\frac{\log T}{\log\log T}\right) \exp\left((\log\log 3n+9) \sum_{j=\lfloor \log\log 3n \rfloor+10}^{\lfloor \frac{\log T}{\log\log T} \rfloor} \frac{1}{j}\right) \\
&\quad \times (\log\log 3n+10)^{\lfloor \log\log 3n \rfloor+9} \\
&\leq (\log T)^{-k} \exp\left(C_{30}\frac{\log T}{\log\log T}\right),
\end{aligned} \tag{148}$$

where we have used (75) and that $n \leq T^M$.

Putting the estimates of (144)–(148) in (143) we obtain

Proposition 4 *Assume RH. Suppose that $T \geq T_0$, $2 \leq n \leq T^M$, where we will take $M > 1$ to be a suitable fixed number. Let $k \in \mathbb{N}$, $0 \leq k \leq \lfloor \frac{\log T}{\log\log T} \rfloor$ and $b = -2$ or 2. Let $\varrho = \frac{1}{2} + i\upsilon$ run through the zeros of $Z_1(s)$. We have*

$$\sum_{\frac{T}{2}<\upsilon\leq T} n^{-\varrho}\left(\frac{\chi'}{\chi}(\varrho+b)\right)^{-k} = \frac{(-1)^k}{2\pi n}\int_{\frac{T}{2}}^{T}\sum_{\ell=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor} 2^{\ell}\left(\frac{2\lambda^{(\ell)}(n)}{(\log\frac{t}{2\pi})^{k+\ell+1}} - \frac{\Lambda^{(\ell+1)}(n)}{(\log\frac{t}{2\pi})^{k+\ell}}\right)dt$$
$$+ O\left((\log T)^{-k}\exp\left(\frac{C_{31}\log T\log\log\log T}{\log\log T}\right)\right). \tag{149}$$

From this result we deduce the estimate for (125) to be used in this section.

Corollary 1 *Under the conditions of Proposition 4, with $2 \leq p^a \leq T^M$, where p is a prime number and $a \in \mathbb{Z}^+$, we have*

$$\sum_{\frac{T}{2}<\upsilon\leq T}(p^a)^{-\varrho} = \frac{T\log p}{4\pi p^a}\left(\left(1+\frac{2\log p}{\log T}\right)^a - 2\right)\cdot\left(1+O\left(\frac{1}{\log T}\right)\right)$$
$$+ O\left(\exp\left(\frac{C_{32}\log T\log\log\log T}{\log\log T}\right)\right). \tag{150}$$

Proof We write (149) with $k = 0$. By (70)–(72), the last term of the summation is with $\ell = \min(a-1, \lfloor \frac{\log T}{\log\log T}\rfloor)$. Then we rearrange the terms of the summation with respect to the power of $\log\frac{t}{2\pi}$. Using the values in (70) and (71), in the case $a \leq \lfloor \frac{\log T}{\log\log T}\rfloor + 1$ the main term coming from (149) reduces to

$$\frac{\log p}{2\pi p^a}\left[-T + \int_{\frac{T}{2}}^{T}\left(1+\frac{2\log p}{\log\frac{t}{2\pi}}\right)^a dt\right],$$

from which it is straightforward to get to (150). The case $a > \lfloor \frac{\log T}{\log\log T}\rfloor + 1$ differs from the previous case within an error

$$\ll \frac{T\log p}{p^a}\sum_{\ell=\lfloor \frac{\log T}{\log\log T}\rfloor}^{a-1}\left[\binom{a-1}{\ell}\left(\frac{2\log p}{\log\frac{T}{2\pi}}\right)^{\ell} + 2\binom{a}{\ell+1}\left(\frac{2\log p}{\log\frac{T}{2\pi}}\right)^{\ell+1}\right.$$
$$\left. + \binom{a-1}{\ell}\left(\frac{2\log p}{\log\frac{T}{2\pi}}\right)^{\ell+1}\right]$$
$$\ll \frac{T\log p}{p^a}\sum_{\ell=\lfloor \frac{\log T}{\log\log T}\rfloor}^{a-1}\binom{a}{\ell+1}\left(\frac{C_{33}\log p}{\log T}\right)^{\ell}.$$

Employing the inequality $\binom{a}{\ell+1} \le \left(\frac{ae}{\ell+1}\right)^{\ell+1}$, which is a simple consequence of Stirling's formula, the last quantity can be simplified to

$$\frac{T\log p^a}{p^a}\sum_{\ell=\lfloor\frac{\log T}{\log\log T}\rfloor}^{a-1}\left(\frac{C_{34}a\log p}{\ell\log T}\right)^{\ell} \ll \frac{T\log p^a}{p^a}\sum_{\ell=\lfloor\frac{\log T}{\log\log T}\rfloor}^{a-1}\left(\frac{C_{35}}{\ell}\right)^{\ell}$$

$$\ll \frac{T\log p^a}{p^a}\left(\frac{C_{36}\log\log T}{\log T}\right)^{\frac{\log T}{\log\log T}} \ll p^{-a}\exp\left(\frac{C_{37}\log T\log\log\log T}{\log\log T}\right),$$

and this is contained in an error term of the type as in (150) (if necessary by replacing C_{32} by a larger constant). □

We now resume the calculation for the correlation of zeta zeros with the zeta maxima on the critical line starting with using (150) in (127). From the sum over $n \le x$ we have

$$-x^{-2}\sum_{n\le x}\Lambda(n)n^2\sum_{\frac{T}{2}<\upsilon\le T}n^{-\varrho} = \frac{T}{4\pi x^2}\sum_{p\le x}p(\log p)^2\left(1-\frac{2\log p}{\log T}\right)\left(1+O\left(\frac{1}{\log T}\right)\right)$$

$$+O\left(\frac{T}{x^2}\sum_{\substack{p^a\le x\\ a\ge 2}}p^a(\log p)^2\left(1+\frac{2\log p}{\log T}\right)^a\right) \tag{151}$$

$$+O\left(\frac{1}{x^2}\exp\left(\frac{C_{38}\log T\log\log\log T}{\log\log T}\right)\sum_{n\le x}\Lambda(n)n^2\right),$$

and from the sum over $x < n \le T^M$ we have

$$-x^2\sum_{x<n\le T^M}\frac{\Lambda(n)}{n^2}\sum_{\frac{T}{2}<\upsilon\le T}n^{-\varrho} = \frac{Tx^2}{4\pi}\sum_{x<p\le T^M}\frac{(\log p)^2}{p^3}\left(1-\frac{2\log p}{\log T}\right)\left(1+O\left(\frac{1}{\log T}\right)\right)$$

$$+O\left(Tx^2\sum_{\substack{x<p^a\le T^M\\ a\ge 2}}\frac{(\log p)^2}{p^{3a}}\left(1+\frac{2\log p}{\log T}\right)^a\right) \tag{152}$$

$$+O\left(x^2\exp\left(\frac{C_{38}\log T\log\log\log T}{\log\log T}\right)\sum_{x<n\le T^M}\frac{\Lambda(n)}{n^2}\right).$$

Now we use some consequences of the prime number theorem obtained by partial summation. The contribution of the last error terms of (151) and (152) is

$$\ll x\exp\left(\frac{C_{38}\log T\log\log\log T}{\log\log T}\right). \tag{153}$$

For the main terms of (151) and (152) we use the following which are obtained from (27) by partial summation:

$$\sum_{p\le x} p\log^2 p = \frac{x^2\log x}{2} - \frac{x^2}{4} + O(x^{\frac{3}{2}}\log^3 2x)$$

$$\sum_{p\le x} p\log^3 p = \frac{x^2\log^2 x}{2} - \frac{x^2\log x}{2} + \frac{x^2}{4} + O(x^{\frac{3}{2}}\log^4 2x)$$

$$\sum_{x<p<T^M} \frac{\log^2 p}{p^3} = \frac{\log x}{2x^2} - \frac{1}{4x^2} + O\left(\frac{\log^3 2x}{x^{\frac{5}{2}}}\right) + O\left(\frac{\log T}{T^{2M}}\right) \tag{154}$$

$$\sum_{x<p<T^M} \frac{\log^3 p}{p^3} = \frac{\log^2 x}{2x^2} + \frac{\log x}{x^2} + \frac{1}{4x^2} + O\left(\frac{\log^4 2x}{x^{\frac{5}{2}}}\right) + O\left(\frac{\log^2 T}{T^{2M}}\right)$$

Then, the main term of the sums on the right-hand side of (127) is seen to be

$$\frac{T}{4\pi}\left(\log x - \frac{2\log^2 x}{\log T} - \frac{1}{2} - \frac{\log x}{\log T}\right)\left(1 + O\left(\frac{1}{\log T}\right)\right) \tag{155}$$
$$+ O(Tx^{-\frac{1}{2}}\log^3 2x) + O(x^2T^{1-2M}\log T).$$

Just by comparing (153) and (155) we see that we are forced to put a restriction such as $x \le T^{1-\epsilon}$ in order to have an asymptotic estimate. It remains to bound the first error terms in (151) and (152). We have

$$\left(1 + \frac{2\log p}{\log T}\right)^a < e^{\frac{2a\log p}{\log T}} \le e^{2M},$$

$$\sum_{\substack{p^a\le x\\ a\ge 2}} p^a(\log p)^2 \le \sum_{a\ge 2} x\log(x^{\frac{1}{a}}) \sum_{p\le x^{\frac{1}{a}}} \log p \ll x\log x \sum_{a\ge 2} \frac{x^{\frac{1}{a}}}{a} \ll x^{\frac{3}{2}}\log 2x,$$

$$\sum_{\substack{x<p^a\le T^M\\ a\ge 2}} \frac{(\log p)^2}{p^{3a}} \ll \sum_{a\ge 2} \frac{\log(x^{\frac{1}{a}})}{x^{3-\frac{2}{a}}} \sum_{p\ge x^{\frac{1}{a}}} \frac{\log p}{p^2} \ll \frac{\log x}{x^3}\sum_{a\ge 2}\frac{x^{\frac{1}{a}}}{a} \ll \frac{\log x}{x^{\frac{5}{2}}}, \tag{156}$$

so that the first error terms in (151) and (152) are

$$\ll Tx^{-\frac{1}{2}}\log 2x. \tag{157}$$

Combining (127), (153), (155), (157), and taking a fixed $M > \frac{4}{3}$, we have

$$\sum_{\gamma} \sum_{\frac{T}{2}<\upsilon\le T} \frac{4x^{i(\gamma-\upsilon)}}{4+(\gamma-\upsilon)^2} \tag{158}$$
$$= \frac{T\log x}{4\pi}\left(1-\frac{2\log x}{\log T}\right) + O(T) + \frac{T\log^2 T}{4\pi x^2}\left(1+O\left(\frac{1}{\log T}\right)\right)$$

for $1 \le x < T^{1-\epsilon}$ as $T \to \infty$.

Now we write the version of (158) for each case with $\frac{T}{2^{k+1}} < \upsilon \le \frac{T}{2^k}$, $k = 0, 1, \ldots, \lfloor \frac{4\log\log T}{\log 2} \rfloor$, and add up the results. For $0 < \upsilon < \frac{T}{\log^4 T}$ we use the trivial estimate

$$\sum_{\gamma} \sum_{0<\upsilon\le \frac{T}{\log^4 T}} \frac{4x^{i(\gamma-\upsilon)}}{4+(\gamma-\upsilon)^2} \ll \frac{T}{\log^4 T}\cdot\log^2 T \ll \frac{T}{\log^2 T}. \tag{159}$$

Then, in the same way as for (24), we discard the terms with $\gamma \notin (0, T]$ within an error of $O(\log^3 T)$. Thus we obtain

Theorem 1 *Assume RH. Let $\frac{1}{2}+i\gamma$ and $\frac{1}{2}+i\upsilon$ run through the critical zeros of $\zeta(s)$ and of $Z_1(s)$, respectively. We have, uniformly for $1 \le x \le T^{1-\epsilon}$,*

$$F_{\zeta,Z_1}(x,T) := \sum_{0<\gamma\le T}\sum_{0<\upsilon\le T} \frac{4x^{i(\gamma-\upsilon)}}{4+(\gamma-\upsilon)^2} \tag{160}$$
$$= \frac{T\log x}{2\pi}\left(1-\frac{2\log x}{\log T}\right) + \frac{T\log^2 T}{2\pi x^2}\left(1+O\left(\frac{1}{\log T}\right)\right) + O(T),$$

as $T \to \infty$, in which case $F_{\zeta,Z_1}(x,T)$ is asymptotically real.

Remark Since the main term of $F_{\zeta,Z_1}(x,T)$ has a sign change in $1 \le x \le T^{1-\epsilon}$, we can't deduce conclusions as Montgomery did using the positivity of $F_{\zeta,\zeta}(x,T)$.

8 Pair Correlation of Zeros of $Z_1(s)$

We shall first derive an explicit formula involving the zeros of $Z_1(s)$, an analogue of (19), which relates a sum over the non-trivial zeros $\varrho = \frac{1}{2}+i\upsilon$ of $Z_1(s)$ to a sum over involving arithmetical functions. We begin by considering the integral

$$\oint_{\mathscr{R}} \frac{Z_1'}{Z_1}(w+\frac{1}{2})K(w,s)x^w\,dw, \quad (x \ge 1), \tag{161}$$

along the rectangular contour $\mathscr{R}$ with vertices at $c \pm iY$, $-U \pm +iY$, with

$$\begin{aligned} &c = \frac{1}{2} + \epsilon_0, \quad (\epsilon_0 > 0 \text{ can be arbitrarily small but fixed}), \\ &U = 2k + 1, \quad (k: \text{ large natural number}), \quad \frac{U}{4} \geq \frac{Y}{2} \geq |t| \\ &w = u + iy,\ u, y \in \mathbb{R}, \quad \sigma \in \left[\frac{3}{2}, \frac{11}{4}\right] \end{aligned} \tag{162}$$

(the restriction on σ is just for convenience, we can take $\sigma > 1 + \epsilon$; later on we will use our explicit formula with $\sigma = \frac{5}{2}$), and

$$K(w, s) := \frac{2\sigma - 1}{\left(w - \left(s - \frac{1}{2}\right)\right)\left(w - \left(\frac{1}{2} - \overline{s}\right)\right)}. \tag{163}$$

The kernel $K(w, s)$ was introduced by Farmer et al. in [5] in their derivation of an explicit formula involving the zeros of $\xi'(s)$, where $\xi(s) = \frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$. The integrand is regular on the contour of integration. By (57) the left vertical side of $\mathscr{R}$ is at a distance $\geq \frac{1}{2}$ from any zero of $Z_1(w + \frac{1}{2})$. By (139), we have

$$\frac{Z_1'}{Z_1}\left(-2k - \frac{1}{2} + iy\right) = \frac{\chi'}{\chi}\left(-2k - \frac{1}{2} + iy\right) - \frac{Z_1'}{Z_1}\left(2k + \frac{3}{2} - iy\right),$$

and by using the last member of (16) we see that

$$\frac{\chi'}{\chi}\left(-2k - \frac{1}{2} + iy\right) = O(\log|w|).$$

From (58) we observe that $\frac{Z_1'}{Z_1}(2k + \frac{3}{2} - iy)$ takes on small values because here the values of $\frac{\zeta'}{\zeta}$ and $\frac{\zeta''}{\zeta}$ are small, and the estimates (16) and (17) are applicable as can be seen through the middle member of (14). Hence we have

$$\frac{Z_1'}{Z_1}\left(w + \frac{1}{2}\right) = O(\log|w|), \quad (\Re\, w = -U = -2k - 1), \tag{164}$$

and

$$\int_{-U-iY}^{-U+iY} \frac{Z_1'}{Z_1}\left(w + \frac{1}{2}\right) K(w, s) x^w dw \ll \frac{Y \log U}{x^U U^2}. \tag{165}$$

The estimate (164) also holds on the part $\Re\, w \leq -1$ of the horizontal sides of $\mathscr{R}$ as can be seen again from (139). On the part $-1 \leq \Re\, w \leq c$ of the horizontal sides, we use (131). Thus the horizontal sides contribute

$$\ll \int_c^{-1} \frac{\log^2 Y}{Y^2} x^u\, du + \int_{-1}^{-Y} \frac{\log Y}{Y^2} x^u\, du + \int_{-Y}^{-U} \frac{\log(u^2+Y^2)}{u^2+Y^2} x^u\, du$$
$$\ll x^c \frac{\log^2 Y}{Y^2} + \frac{\log Y}{Yx} + x^{-Y}\frac{\log Y}{Y}. \tag{166}$$

If we first let $U \to \infty$, and then let $Y \to \infty$, the integrals along the horizontal and left vertical sides will tend to 0. In the region enclosed by $\mathscr{R}$, the poles of the integrand are all simple poles at the points $w = -\frac{1}{2},\ 0,\ \frac{1}{2},\ \frac{1}{2}-\overline{s},\ i\upsilon$ and at the points $\frac{1}{2} - z_\ell,\ \ell \in \mathbb{Z}^+$. Thus, by the residue theorem we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{(c)} \frac{Z_1'}{Z_1}\left(w+\frac{1}{2}\right) K(w,s)x^w\, dw &= -K\left(-\frac{1}{2},s\right)x^{-\frac{1}{2}} + K(0,s) - 2K\left(\frac{1}{2},s\right)x^{\frac{1}{2}} \\ &\quad - \frac{Z_1'}{Z_1}(1-\overline{s})x^{\frac{1}{2}-\overline{s}} \\ &\quad + \sum_{\varrho} K(i\upsilon,s)x^{i\upsilon} + \sum_{\ell=1}^{\infty} K\left(\frac{1}{2}-z_\ell,s\right)x^{\frac{1}{2}-z_\ell},\end{aligned} \tag{167}$$

where $\int_{(c)}$ denotes that the integral is over the whole line $\Re\, w = c$. To simplify this formula, we use the estimates

$$\left|K\left(-\frac{1}{2},s\right)x^{-\frac{1}{2}}\right| = \frac{(2\sigma-1)x^{-\frac{1}{2}}}{|s(1-\overline{s})|} \ll \frac{x^{-\frac{1}{2}}}{1+t^2},$$
$$\left|K\left(\frac{1}{2},s\right)x^{\frac{1}{2}}\right| = \frac{(2\sigma-1)x^{\frac{1}{2}}}{|(1-s)\overline{s}|} \ll \frac{x^{\frac{1}{2}}}{1+t^2},$$
$$|K(0,s)| = \frac{2\sigma-1}{\left|s-\frac{1}{2}\right|^2} \ll \frac{1}{1+t^2},$$

and we majorize the last sum in (167), by using the information given in (57), as

$$\sum_{\ell=1}^{\infty} K\left(\frac{1}{2}-z_\ell,s\right)x^{\frac{1}{2}-z_\ell} \ll \sum_{\ell=1}^{\infty} \frac{x^{\frac{1}{2}-z_\ell}}{\ell^2+t^2} \ll \frac{x^{-\frac{5}{2}}}{1+|t|}.$$

We can now re-write (167) as

$$\begin{aligned}\frac{1}{2\pi i}\int_{(c)} \frac{Z_1'}{Z_1}\left(w+\frac{1}{2}\right) K(w,s)x^w\, dw &= -\sum_{\varrho} \frac{(2\sigma-1)x^{i\upsilon}}{(\sigma-\frac{1}{2})^2+(t-\upsilon)^2} - x^{\frac{1}{2}-\overline{s}}\frac{Z_1'}{Z_1}(1-\overline{s}) \\ &\quad + O\left(\frac{x^{\frac{1}{2}}}{1+|t|^2}\right) + O\left(\frac{x^{-\frac{5}{2}}}{1+|t|}\right).\end{aligned} \tag{168}$$

For use in the evaluation of the line integral in (168), we expand the expression in (58) for $\frac{Z_1'}{Z_1}(z)$, where $\Re\, z = c+\frac{1}{2} = 1+\epsilon_0 > 1$, $\Im\, z \geq A$ and A is some sufficiently large constant, by keeping the factors $\frac{\chi'}{\chi}(z)$ intact. Assuming RH, we have

$$\frac{Z_1'}{Z_1}(z) = -\sum_{k=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor} \left(\frac{-2}{\frac{\chi'}{\chi}(z)}\right)^k \sum_{m=1}^{\infty} \frac{\Lambda^{(k+1)}(m)}{m^z} + \sum_{k=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor} \left(\frac{-2}{\frac{\chi'}{\chi}(z)}\right)^{k+1} \sum_{m=1}^{\infty} \frac{\lambda^{(k)}(m)}{m^z} + O\left(\exp\left(\left(C_{39}(\epsilon_0) - \log\log|z|\right)\frac{\log T}{\log\log T}\right)\right) + O\left(\frac{1}{|z|\log|z|}\right). \tag{169}$$

Next, we split the integral into several parts as

$$\frac{1}{2\pi i}\int_{(c)} \frac{Z_1'}{Z_1}\left(w+\frac{1}{2}\right) K(w,s)x^w\, dw \tag{170}$$

$$= -\sum_{k=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor} (-2)^k \sum_{m=1}^{\infty} \frac{\Lambda^{(k+1)}(m)}{m^{\frac{1}{2}}} \frac{1}{2\pi i}\int_{(c)} \frac{K(w,s)}{\left(\frac{\chi'}{\chi}\left(w+\frac{1}{2}\right)\right)^k}\left(\frac{x}{m}\right)^w dw$$

$$+ \sum_{k=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor} (-2)^{k+1} \sum_{m=1}^{\infty} \frac{\lambda^{(k)}(m)}{m^{\frac{1}{2}}} \frac{1}{2\pi i}\int_{(c)} \frac{K(w,s)}{\left(\frac{\chi'}{\chi}\left(w+\frac{1}{2}\right)\right)^{k+1}}\left(\frac{x}{m}\right)^w dw$$

$$+ \sum_{k=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor} (-2)^k \sum_{m=1}^{\infty} \frac{\Lambda^{(k+1)}(m)}{m^{\frac{1}{2}}} \frac{1}{2\pi i}\int_{c-iA}^{c+iA} \frac{K(w,s)}{\left(\frac{\chi'}{\chi}\left(w+\frac{1}{2}\right)\right)^k}\left(\frac{x}{m}\right)^w dw$$

$$- \sum_{k=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor} (-2)^{k+1} \sum_{m=1}^{\infty} \frac{\lambda^{(k)}(m)}{m^{\frac{1}{2}}} \frac{1}{2\pi i}\int_{c-iA}^{c+iA} \frac{K(w,s)}{\left(\frac{\chi'}{\chi}\left(w+\frac{1}{2}\right)\right)^{k+1}}\left(\frac{x}{m}\right)^w dw$$

$$+ O\left(x^c \int_{-iA}^{iA} \left|\frac{Z_1'}{Z_1}\left(c+\frac{1}{2}+iy\right)K(c+iy,s)\right| dy\right)$$

$$+ O\left(x^c \int_{|y|\geq A} |K(c+iy,s)| \left(\exp\left(\frac{(C_{39}(\epsilon_0) - \log\log|y|)\log T}{\log\log T}\right)\right.\right.$$

$$\left.\left. + \frac{1}{|y|\log|y|}\right) dy\right) = I_1 + I_2 + \cdots + I_6, \text{ say.}$$

We will work with large $|t| \in [2A, T]$; later on T will be a parameter which tends to ∞. By using

$$K(c+iy,s) \ll \frac{1}{1+(y-t)^2}, \tag{171}$$

it is easily seen that

$$I_5 \ll \frac{x^c}{1+t^2}, \quad I_6 \ll \frac{x^c}{1+|t|}\exp\left(\frac{C_{40}(\epsilon_0)\log T}{\log\log T}\right). \tag{172}$$

Now note that since the zero of $\chi'(s)$ is in $(\frac{5}{4}, 2)$, if we take $\epsilon_0 < \frac{1}{5}$, then by (16) we have $\left|\frac{\chi'}{\chi}(w+\frac{1}{2})\right|^{-1} \le C_{41}$, so that

$$\begin{aligned} I_3 &\ll x^c \sum_{k=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor} (2C_{41})^k \sum_{m=1}^{\infty} \frac{\Lambda^{(k+1)}(m)}{m^{c+\frac{1}{2}}} \int_{-A}^{A} |k(c+iy,s)|\,dy \\ &\ll \frac{x^c}{1+t^2} \sum_{k=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor} \left(2C_{41}\frac{\zeta'}{\zeta}\left(c+\frac{1}{2}\right)\right)^{k+1} \ll \frac{x^c}{1+t^2} C_{42}(\epsilon_0)^{\frac{\log T}{\log\log T}}, \end{aligned} \tag{173}$$

and similarly I_4 satisfies the same bound. Thus

$$\frac{1}{2\pi i}\int_{(c)} \frac{Z_1'}{Z_1}\left(w+\frac{1}{2}\right) K(w,s)x^w\,dw = I_1 + I_2 + O\left(\frac{x^c}{|t|}\exp\left(\frac{C_{43}(\epsilon_0)\log T}{\log\log T}\right)\right). \tag{174}$$

The evaluation of I_1 and I_2 splits into two cases according as $m \le x$ or not. Consider first the situation when $m \le x$ in which case the line of integration $\Re\, w = c$ will be pulled to the left. Let $U = 2n+1$. Then, by (14), we see that $\frac{\chi'}{\chi}(-U+\frac{1}{2}+it) \gg \log U$. Between the lines $\Re\, w = c$ and $\Re\, w = -U$ the integrands have poles coming from the zeros of the $\frac{\chi'}{\chi}$ factor at $w = \kappa_\ell - \frac{1}{2}$, $\ell = 1, 2, \ldots, n$, at $w = \pm i\tau$ by (41) and (53), and a pole arising from $K(w,s)$ at $w = \frac{1}{2} - \bar{s}$. By the residue theorem we write

$$\begin{aligned} &\frac{1}{2\pi i}\int_{c-iY}^{c+iY} \frac{K(w,s)}{\left(\frac{\chi'}{\chi}\left(w+\frac{1}{2}\right)\right)^k}\left(\frac{x}{m}\right)^w dw \\ &\quad = \text{Residues} + \left(\int_{c-iY}^{-U-iY} + \int_{-U-iY}^{-U+iY} + \int_{-U+iY}^{c+iY}\right) \frac{K(w,s)}{\left(\frac{\chi'}{\chi}\left(w+\frac{1}{2}\right)\right)^k}\left(\frac{x}{m}\right)^w dw \end{aligned}$$

with the parameters having the same specifications as in (162). The last three integrals all tend to 0 as $n \to \infty$ (i.e. $-U \to -\infty$) and $Y \to \infty$ as in (165)–(166). Now, applying Cauchy's estimate on a disk of radius ϵ_1 centred at $\kappa_\ell - \frac{1}{2}$, for $k \ge 1$ we have

$$\begin{aligned}
&\operatorname*{Res}_{w=\kappa_\ell-1/2}\left\{\frac{K(w,s)}{\left(\frac{\chi'}{\chi}(w+1/2)\right)^k}\left(\frac{x}{m}\right)^w\right\}\\
&=\frac{1}{(k-1)!}\frac{d^{k-1}}{dw^{k-1}}\left\{K(w,s)\left(\frac{w-\left(\kappa_\ell-\frac{1}{2}\right)}{\frac{\chi'}{\chi}\left(w+\frac{1}{2}\right)}\right)^k\left(\frac{x}{m}\right)^w\right\}_{w=\kappa_\ell-\frac{1}{2}}\\
&\ll\frac{C_{44}(\epsilon_1)^k\left(\frac{x}{m}\right)^{\kappa_\ell-\frac{1}{2}+\epsilon_1}}{(\ell+|t|)^2}.
\end{aligned}$$

Summing over all ℓ, we can include the contribution from all of these residues in

$$\frac{C_{45}(\epsilon_1)^k}{1+|t|}\left(\frac{x}{m}\right)^{\kappa_1-\frac{1}{2}+\epsilon_1}$$

Similarly we see that

$$\operatorname*{Res}_{w=\pm i\tau}\left\{\frac{K(w,s)}{\left(\frac{\chi'}{\chi}(w+1/2)\right)^k}\left(\frac{x}{m}\right)^w\right\}\ll\frac{C_{46}(\epsilon_1)^k\left(\frac{x}{m}\right)^{\epsilon_1}}{1+t^2},\quad(k\geq 1),$$

and

$$\operatorname*{Res}_{w=\frac{1}{2}-\overline{s}}\left\{\frac{K(w,s)}{\left(\frac{\chi'}{\chi}\left(w+\frac{1}{2}\right)\right)^k}\left(\frac{x}{m}\right)^w\right\}=-\left(\frac{\chi'}{\chi}(1-\overline{s})\right)^{-k}\left(\frac{x}{m}\right)^{\frac{1}{2}-\overline{s}}$$

(for $\sigma\in[\frac{3}{2},\frac{11}{4}]$ this pole is away from the zeros of $\frac{\chi'}{\chi}(1-\overline{s})$). Collecting the foregoing estimates, we obtain (with no error term if $k=0$)

$$\begin{aligned}
\frac{1}{2\pi i}\int_{(c)}\frac{K(w,s)}{\left(\frac{\chi'}{\chi}\left(w+\frac{1}{2}\right)\right)^k}\left(\frac{x}{m}\right)^w dw&=-\left(\frac{\chi'}{\chi}(1-\overline{s})\right)^{-k}\left(\frac{x}{m}\right)^{\frac{1}{2}-\overline{s}}\\
&\quad+O\left(\frac{C_{45}(\epsilon_1)^k}{1+|t|}\left(\frac{x}{m}\right)^{\kappa_1-\frac{1}{2}+\epsilon_1}\right)\\
&\quad+O\left(\frac{C_{46}(\epsilon_1)^k}{1+t^2}\left(\frac{x}{m}\right)^{\epsilon_1}\right),\quad(m\leq x).
\end{aligned}\tag{175}$$

In case $m>x$, the calculation is done similarly, but now the line of integration is pulled to the right to $\Re\, w=2n$, followed by letting the integer $n\to\infty$. The poles encountered are at $w=\frac{1}{2}-\kappa_\ell$, $\ell=1,2,\ldots$ from the $\frac{\chi'}{\chi}$ factor, and at $w=s-\frac{1}{2}$

from $K(w,s)$. The result is (with no error term if $k=0$)

$$\frac{1}{2\pi i}\int_{(c)}\frac{K(w,s)}{\left(\frac{\chi'}{\chi}\left(w+\frac{1}{2}\right)\right)^k}\left(\frac{x}{m}\right)^w dw = -\left(\frac{\chi'}{\chi}(s)\right)^{-k}\left(\frac{x}{m}\right)^{s-\frac{1}{2}} + O\left(\frac{C_{47}(\epsilon_1)^k}{1+|t|}\left(\frac{x}{m}\right)^{\frac{1}{2}-\kappa_1-\epsilon_1}\right), \quad (m>x). \tag{176}$$

Writing, for brevity,

$$\mathfrak{L}(m,s) := \sum_{k=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor}(-2)^k\left(\Lambda^{(k+1)}(m)\left(\frac{\chi'}{\chi}(s)\right)^{-k} + 2\lambda^{(k)}(m)\left(\frac{\chi'}{\chi}(s)\right)^{-k-1}\right), \tag{177}$$

we see from (175)–(177) that

$$\begin{aligned} I_1+I_2 = x^{-\frac{1}{2}}&\left(\sum_{m\le x}\mathfrak{L}(m,1-\bar{s})\left(\frac{x}{m}\right)^{1-\bar{s}} + \sum_{m>x}\mathfrak{L}(m,s)\left(\frac{x}{m}\right)^{s}\right)\\ &+ O\left(\frac{x^{\kappa_1-\frac{1}{2}+\epsilon_1}}{1+|t|}\sum_{k=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor}C_{48}(\epsilon_1)^k\sum_{m\le x}\frac{\Lambda^{(k+1)}(m)+\lambda^{(k)}(m)}{m^{\kappa_1+\epsilon_1}}\right)\\ &+ O\left(\frac{x^{\epsilon_1}}{1+t^2}\sum_{k=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor}C_{49}(\epsilon_1)^k\sum_{m\le x}\frac{\Lambda^{(k+1)}(m)+\lambda^{(k)}(m)}{m^{\frac{1}{2}+\epsilon_1}}\right)\\ &+ O\left(\frac{x^{\frac{1}{2}-\kappa_1-\epsilon_1}}{1+|t|}\sum_{k=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor}C_{50}(\epsilon_1)^k\sum_{m>x}\frac{\Lambda^{(k+1)}(m)+\lambda^{(k)}(m)}{m^{1-\kappa_1-\epsilon_1}}\right). \end{aligned}$$

We can put a bound on the error terms by first replacing $m^{-\kappa_1-\epsilon_1}$ with $m^{-1-\epsilon_1}x^{1-\kappa_1}$ in the first series over $m\le x$, and $m^{-\frac{1}{2}-\epsilon_1}$ by $x^{\frac{1}{2}}m^{-1-\epsilon_1}$ in the second series over $m\le x$, and $m^{-1+\kappa_1+\epsilon_1}$ with $m^{-1-\epsilon_1}x^{\kappa_1+2\epsilon_1}$ for $m>x$, and then using (136) to bound the resulting Dirichlet series, so that we have

$$\begin{aligned} I_1+I_2 = x^{-\frac{1}{2}}&\left(\sum_{m\le x}\mathfrak{L}(m,1-\bar{s})\left(\frac{x}{m}\right)^{1-\bar{s}} + \sum_{m>x}\mathfrak{L}(m,s)\left(\frac{x}{m}\right)^{s}\right)\\ &+ O\left(\frac{x^{\frac{1}{2}+\epsilon_1}}{1+|t|}\exp\left(\frac{C_{51}(\epsilon_1)\log T}{\log\log T}\right)\right). \end{aligned} \tag{178}$$

The last step for completing the derivation of the explicit formula for $Z_1(s)$ is replacing the term $-x^{\frac{1}{2}-\overline{s}}\frac{Z_1'}{Z_1}(1-\overline{s})$ in (168) with

$$x^{\frac{1}{2}-\overline{s}}\frac{Z_1'}{Z_1}(\overline{s})-x^{\frac{1}{2}-\overline{s}}\frac{\chi'}{\chi}(\overline{s})=x^{\frac{1}{2}-\overline{s}}\log\frac{|t|}{2\pi}+O_\sigma(x^{\frac{1}{2}-\sigma}), \tag{179}$$

by using (139).

Combining (168), (174), (177)–(179) we obtain

Proposition 5 (Explicit Formula; Assuming RH) *For $s=\sigma+it$ with $\frac{3}{2}\le\sigma\le\frac{11}{4}$, $|t|\in[2A,T]$ with A a sufficiently large constant, and for any arbitrarily small fixed $\epsilon>0$, we have, as $T\to\infty$,*

$$\begin{aligned}\sum_{\varrho}\frac{(2\sigma-1)x^{i\nu}}{(\sigma-\frac{1}{2})^2+(\nu-t)^2}&=-x^{-\frac{1}{2}}\left(\sum_{m\le x}\mathfrak{L}(m,1-\overline{s})\left(\frac{x}{m}\right)^{1-\overline{s}}+\sum_{m>x}\mathfrak{L}(m,s)\left(\frac{x}{m}\right)^{s}\right)\\&\quad+x^{\frac{1}{2}-\overline{s}}\log\frac{|t|}{2\pi}+O\left(\frac{x^{\frac{1}{2}+\epsilon}}{|t|}\exp\left(\frac{C_{52}(\epsilon)\log T}{\log\log T}\right)\right)\\&\quad+O_\sigma(x^{\frac{1}{2}-\sigma}),\end{aligned} \tag{180}$$

where $\mathfrak{L}(m,s)$ was defined in (177).

In (180) we take $\sigma=\frac{5}{2}$ and sum both sides over $t=\tilde{\upsilon}\in(\frac{T}{2},T]$, where $\frac{1}{2}+i\tilde{\upsilon}$ runs through the non-trivial zeros of $Z_1(s)$. In view of (55) and (1) we have

$$\begin{aligned}&\sum_{\upsilon}\sum_{\frac{T}{2}<\tilde{\upsilon}\le T}\frac{4x^{i(\upsilon-\tilde{\upsilon})}}{4+(\upsilon-\tilde{\upsilon})^2} \qquad (181)\\&=-x^{-2}\sum_{m\le x}\sum_{\frac{T}{2}<\tilde{\upsilon}\le T}\mathfrak{L}\left(m,-\frac{3}{2}+i\tilde{\upsilon}\right)m^{\frac{3}{2}-i\tilde{\upsilon}}-x^2\sum_{m>x}\sum_{\frac{T}{2}<\tilde{\upsilon}\le T}\mathfrak{L}\left(m,\frac{5}{2}+i\tilde{\upsilon}\right)m^{-\frac{5}{2}-i\tilde{\upsilon}}\\&\quad+\frac{T\log^2T}{4\pi x^2}\left(1+O\left(\frac{1}{\log T}\right)\right)+O\left(x^{\frac{1}{2}+\epsilon}\exp\left(\frac{C_{53}(\epsilon)\log T}{\log\log T}\right)\right).\end{aligned}$$

We do not hope to get an asymptotic estimate for $x>T$, so just as we did in connection with (126), let us assume $x\le T^M$ and first discard the terms with $m>T^M$ for some fixed $M>\frac{4}{3}$. Upon using (16) and trivially estimating the sum over $\tilde{\upsilon}$ by (55), we have

$$x^2\sum_{m>T^M}\sum_{\frac{T}{2}<\tilde{\upsilon}\le T}\mathfrak{L}\left(m,\frac{5}{2}+i\tilde{\upsilon}\right)m^{-\frac{5}{2}-i\tilde{\upsilon}}$$

$$= x^2 \sum_{k=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} (-2)^k \sum_{m>T^M} m^{-\frac{5}{2}} \Bigg[\sum_{\frac{T}{2}<\tilde{\upsilon}\le T} \Lambda^{(k+1)}(m) \left(\frac{\chi'}{\chi}\left(\frac{5}{2}+i\tilde{\upsilon}\right)\right)^{-k} m^{-i\tilde{\upsilon}}$$

$$+ \sum_{\frac{T}{2}<\tilde{\upsilon}\le T} 2\lambda^{(k)}(m) \left(\frac{\chi'}{\chi}\left(\frac{5}{2}+i\tilde{\upsilon}\right)\right)^{-(k+1)} m^{-i\tilde{\upsilon}} \Bigg]$$

$$\ll x^2 T \log T \Bigg[\sum_{k=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} \left(\frac{2}{\log \frac{T}{2\pi}} \right)^k \sum_{m>T^M} \frac{\Lambda^{(k+1)}(m)}{m^{\frac{5}{2}}}$$

$$+ \sum_{k=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} \left(\frac{2}{\log \frac{T}{2\pi}} \right)^{k+1} \sum_{m>T^M} \frac{\lambda^{(k)}(m)}{m^{\frac{5}{2}}} \Bigg].$$

By (65) and (66) we see

$$\sum_{m>T^M} \frac{\Lambda^{(k+1)}(m)}{m^{\frac{5}{2}}} \ll \frac{(\log T^M)^{k+1}}{T^{\frac{3}{2}M}}, \quad \sum_{m>T^M} \frac{\lambda^{(k)}(m)}{m^{\frac{5}{2}}} \ll \frac{(\log T^M)^{k+2}}{T^{\frac{3}{2}M}},$$

so that

$$x^2 \sum_{m>T^M} \sum_{\frac{T}{2}<\tilde{\upsilon}\le T} \mathfrak{L}\left(m, \frac{5}{2}+i\tilde{\upsilon}\right) m^{-\frac{5}{2}-i\tilde{\upsilon}} \ll x^2 T^{1-\frac{3M}{2}} \exp\left(\frac{C_{54}\log T}{\log\log T}\right). \tag{182}$$

For the $m \le T^M$ terms we use Proposition 4. We first note that the contribution of the error term of (149) to (181) is

$$x \exp\left(\frac{C_{55}\log T \log\log\log T}{\log\log T}\right), \tag{183}$$

by a calculation similar to that leading to (182). Putting these together we have

$$\sum_{\upsilon} \sum_{\frac{T}{2}<\tilde{\upsilon}\le T} \frac{4x^{i(\upsilon-\tilde{\upsilon})}}{4+(\upsilon-\tilde{\upsilon})^2} \tag{184}$$

$$= \frac{U(0,0;x) - 2U(0,1;x) + U(1,1;x)}{2\pi x^2}$$

$$+ \frac{x^2}{2\pi}\left(\tilde{U}(0,0;x) - 2\tilde{U}(0,1;x) + \tilde{U}(1,1;x)\right)$$

$$+ \frac{T\log^2 T}{4\pi x^2}\left(1 + O\left(\frac{1}{\log T}\right)\right) + O\left(x\exp\left(\frac{C_{56}\log T\log\log\log T}{\log\log T}\right)\right),$$

where, for ι_1 and ι_2 taking on the values 0 or 1 and $\Delta(k,\iota;n)$ defined in (77)–(78),

$$U(\iota_1,\iota_2;x) := \sum_{k_1,k_2=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor} \int_{\frac{T}{2}}^{T} \left(\frac{2}{\log\frac{t}{2\pi}}\right)^{k_1+k_2+\iota_1+\iota_2} dt \times \sum_{m\le x} \Delta(k_1+[\iota_1=0],\iota_1;m)\Delta(k_2+[\iota_2=0],\iota_2;m)m, \tag{185}$$

$$\tilde{U}(\iota_1,\iota_2;x) := \sum_{k_1,k_2=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor} \int_{\frac{T}{2}}^{T} \left(\frac{2}{\log\frac{t}{2\pi}}\right)^{k_1+k_2+\iota_1+\iota_2} dt \times \sum_{x<m\le T^M} \frac{\Delta(k_1+[\iota_1=0],\iota_1;m)\Delta(k_2+[\iota_2=0],\iota_2;m)}{m^3}. \tag{186}$$

Formulas (122)–(123) in Proposition 3 are for the evaluation of the sums over m in (185)–(186), but for this application we need to extend the range of the last sum of (186) to all $m > x$. We have, using (65) and (66),

$$\sum_{m>T^M} \frac{\Delta(k_1+[\iota_1=0],\iota_1;m)\Delta(k_2+[\iota_2=0],\iota_2;m)}{m^3} \le \sum_{m>T^M} \frac{(\log m)^{k_1+k_2+\iota_1+\iota_2+2}}{m^3} \ll \frac{(\log T^M)^{k_1+k_2+\iota_1+\iota_2+1}}{T^{2M}}. \tag{187}$$

When plugged in (186) this brings an error

$$\ll T^{1-2M}\log T \sum_{k_1,k_2=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor} \left(\frac{2M\log T}{\log\frac{T}{4\pi}}\right)^{k_1+k_2+\iota_1+\iota_2} \ll T^{1-2M}\log T \sum_{k_1,k_2=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor} (3M)^{k_1+k_2} \ll T^{1-2M}(\log T)(3M)^{\frac{3\log T}{\log\log T}} \ll T^{1-\frac{3}{2}M}, \tag{188}$$

which is very small for $M > \frac{4}{3}$. In applying (122)–(123), the error terms of these formulas give rise to error terms of the form

$$\ll T \sum_{k_1,k_2=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor} \frac{1}{\max(k_1,k_2)!}\left(\frac{2C_{18}\log x}{\log\frac{T}{4\pi}}\right)^{k_1+k_2+\ell}, \quad (\ell=0,1,2),$$

and recalling that $x \le T^M$, we see that these error terms can be bounded as

$$T\sum_{k_1,k_2=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}\frac{C_{57}^{k_1+k_2}}{\max(k_1,k_2)!}\ll T\sum_{k_2=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}\frac{C_{57}^{k_2}}{k_2!}\sum_{k_1\le k_2}C_{57}^{k_1}\tag{189}$$
$$\ll T\sum_{k_2=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}\frac{C_{58}^{k_2}}{k_2!}\le e^{C_{58}}T=C_{59}T.$$

Now, going back to (184) we have

$$\sum_{\upsilon}\sum_{\frac{T}{2}<\tilde{\upsilon}\le T}\frac{4x^{i(\upsilon-\tilde{\upsilon})}}{4+(\upsilon-\tilde{\upsilon})^2}\tag{190}$$
$$=\frac{1}{2\pi}\sum_{k_1,k_2=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}\Bigg[\frac{P(k_1+1,k_2+1,0,0)(\log x)^{k_1+k_2+1}}{(k_1+k_2+1)!}\int_{\frac{T}{2}}^{T}\left(\frac{2}{\log\frac{t}{2\pi}}\right)^{k_1+k_2}dt$$
$$-2\frac{P(k_1+1,k_2,0,1)(\log x)^{k_1+k_2+2}}{(k_1+k_2+2)!}\int_{\frac{T}{2}}^{T}\left(\frac{2}{\log\frac{t}{2\pi}}\right)^{k_1+k_2+1}dt$$
$$+\frac{P(k_1,k_2,1,1)(\log x)^{k_1+k_2+3}}{(k_1+k_2+3)!}\int_{\frac{T}{2}}^{T}\left(\frac{2}{\log\frac{t}{2\pi}}\right)^{k_1+k_2+2}dt\Bigg]$$
$$+\frac{T\log^2 T}{4\pi x^2}\left(1+O\left(\frac{1}{\log T}\right)\right)+O\left(x\exp\left(\frac{C_{56}\log T\log\log\log T}{\log\log T}\right)\right)+O(T).$$

Here we plug in the values of $P(k_1,k_2,\iota_1,\iota_2)$ given in (100) and we use

$$\int_{\frac{T}{2}}^{T}\left(\frac{2}{\log\frac{t}{2\pi}}\right)^{\ell}dt=\frac{T}{2}\left(\frac{2}{\log\frac{T}{2\pi}}\right)^{\ell}\left(1+O\left(\frac{\ell}{\log T}\right)\right),\quad\left(\ell\in\mathbb{N},\,\ell\ll\frac{\log T}{\log\log T}\right),\tag{191}$$

and upon organizing the terms we obtain

$$\sum_{\upsilon}\sum_{\frac{T}{2}<\tilde{\upsilon}\le T}\frac{4x^{i(\upsilon-\tilde{\upsilon})}}{4+(\upsilon-\tilde{\upsilon})^2}\tag{192}$$
$$=\frac{T\log x}{4\pi}\left[1-4\frac{\log x}{\log\frac{T}{2\pi}}+\sum_{k=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}\frac{2\cdot k!}{(2k+2)!}\left(\frac{2\log x}{\log\frac{T}{2\pi}}\right)^{2k+2}\right]\cdot\left(1+O\left(\frac{\log\log T}{\log T}\right)\right)$$
$$+\frac{T\log^2 T}{4\pi x^2}\left(1+O\left(\frac{1}{\log T}\right)\right)+O\left(x\exp\left(\frac{C_{56}\log T\log\log\log T}{\log\log T}\right)\right)+O(T).$$

We would like to extend the sum over k to an infinite sum. Observe that for $x = T$ we no longer have an asymptotic estimate, so we take $1 \le x \le T^{1-\epsilon}$. Then $0 \le \frac{2\log x}{\log \frac{T}{2\pi}} < 2$, and with $K = \lceil \frac{\log T}{\log\log T} \rceil$ we have

$$\begin{aligned}\sum_{k=K}^{\infty} \frac{k!}{(2k+2)!} 2^{2k+2} &\le \sum_{k=K}^{\infty} \left(\frac{1}{k+1}\right)^{k+2} 4^{k+1} \le \frac{1}{K} \sum_{k=K}^{\infty} \left(\frac{4}{k+1}\right)^{k+1} \\ &\le \frac{1}{K} \left(\frac{4}{K+1}\right)^{K+1} \left(1 + \frac{4}{K+1} + \left(\frac{4}{K+1}\right)^2 + \cdots\right) \\ &= \frac{1}{K} \left(\frac{4}{K+1}\right)^{K+1} \frac{1}{1 - \frac{4}{K+1}} \\ &\ll \frac{\log\log T}{\log T} \left(\frac{4\log\log T}{\log T}\right)^{\frac{\log T}{\log\log T}} \ll T^{-1+\epsilon}. \end{aligned} \tag{193}$$

Hence (192) can be re-written as

$$\begin{aligned}&\sum_{\upsilon} \sum_{\frac{T}{2} < \tilde{\upsilon} \le T} \frac{4x^{i(\upsilon-\tilde{\upsilon})}}{4 + (\upsilon - \tilde{\upsilon})^2} \\ &= \frac{T\log x}{4\pi} \left[1 - 4\frac{\log x}{\log \frac{T}{2\pi}} + \sum_{k=0}^{\infty} \frac{2 \cdot k!}{(2k+2)!} \left(\frac{2\log x}{\log \frac{T}{2\pi}}\right)^{2k+2}\right] \cdot \left(1 + O\left(\frac{\log\log T}{\log T}\right)\right) \\ &+ \frac{T\log^2 T}{4\pi x^2} \left(1 + O\left(\frac{1}{\log T}\right)\right) + O(T), \qquad (1 \le x \le T^{1-\epsilon}). \end{aligned} \tag{194}$$

Now we do the same process as after (155), and obtain

Theorem 2 *Assume RH. Let* $\frac{1}{2} + i\upsilon$ *and* $\frac{1}{2} + i\tilde{\upsilon}$ *run through the critical zeros of* $Z_1(s)$. *We have, as* $T \to \infty$, *uniformly for* $1 \le x \le T^{1-\epsilon}$,

$$\begin{aligned}F_{Z_1,Z_1}(x,T) &:= \sum_{0 < \upsilon, \tilde{\upsilon} \le T} \frac{4x^{i(\upsilon-\tilde{\upsilon})}}{4 + (\upsilon - \tilde{\upsilon})^2} \\ &= \frac{T\log x}{2\pi} \left[1 - 4\frac{\log x}{\log \frac{T}{2\pi}} + \sum_{k=0}^{\infty} \frac{2 \cdot k!}{(2k+2)!} \left(\frac{2\log x}{\log \frac{T}{2\pi}}\right)^{2k+2}\right] \cdot \left(1 + O\left(\frac{\log\log T}{\log T}\right)\right) \\ &+ \frac{T\log^2 T}{2\pi x^2} \left(1 + O\left(\frac{1}{\log T}\right)\right) + O(T). \end{aligned} \tag{195}$$

The formula (3) also holds for $F_{Z_1,Z_1}(x,T)$.

Remark The asymptotic value in (195) is the same as the value found by Farmer et al. [5] for the pair correlation of the zeros of the derivative of the Riemann

ξ-function. In [5] an explanation of why the leading order terms of the pair correlation functions for ξ' and Z' should be the same was given. Since the leading order of the correlations are the same, the same conclusions can be drawn from Theorem 3 as Farmer, Gonek and Lee's results. Thus we can say that, assuming RH, a positive proportion of the gaps between consecutive zeros of $Z_1(s)$ are smaller than 0.89661 times the average spacing $\left(\sim \dfrac{2\pi}{\log T}\right)$, and more than 85.838% of the zeros of $Z_1(s)$ are simple. For the proofs of these results and references to earlier related results we refer the reader to [5].

We now define, suppressing the dependence on T in the notation,

$$\mathscr{F}(\alpha) := \left(\frac{T}{\pi}\log T\right)^{-1}\left[F_{\zeta,\zeta}\left(\left(\frac{T}{2\pi}\right)^{\alpha},T\right)+2F_{\zeta,Z_1}\left(\left(\frac{T}{2\pi}\right)^{\alpha},T\right)+F_{Z_1,Z_1}\left(\left(\frac{T}{2\pi}\right)^{\alpha},T\right)\right], \tag{196}$$

so that

$$\mathscr{F}(\alpha) = \left(\frac{T}{\pi}\log T\right)^{-1} \sum_{\substack{0<t_1,t_2\le T\\ \zeta\cdot Z_1(\frac{1}{2}+it_j)=0,\,(j=1,2)}} \frac{4}{4+(t_1-t_2)^2}\left(\frac{T}{2\pi}\right)^{i\alpha(t_1-t_2)}. \tag{197}$$

Note that we have taken $x = \dfrac{T}{2\pi}$ because of the powers in the infinite series in (195). This won't make any difference in the asymptotics for $F_{\zeta,\zeta}$ and F_{ζ,Z_1}. Then we know from (4) and Theorems 1 and 2 that, for $0 \le |\alpha| \le 1-\epsilon$ with any arbitrarily small but fixed ϵ,

$$\mathscr{F}(\alpha) \sim 2|\alpha| - 4|\alpha|^2 + \frac{1}{2}\sum_{m=1}^{\infty}\frac{(m-1)!}{(2m)!}(2|\alpha|)^{2m+1} + 2T^{-2|\alpha|}\log T, \tag{198}$$

as $T\to\infty$.

In order to show the existence of small gaps between zeros of $\zeta\cdot Z_1$ along with a value for the size of such gaps, we follow the method developed by Montgomery [17] and later on also used in [5]. So we will take $h(u)$ to be a minorant of the characteristic function of the interval $[-1,1]$ such that the domain of its Fourier transform $\hat{h}(\alpha)$ includes $[-1,1]$ and is non-negative everywhere. Furthermore, to minimize losses we will require $h(0)=1$. Montgomery [17] used the pair $h(u) = \max(1-|u|,0)$, $\hat{h}(\alpha) = \left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2$; in [5], $h(u) = \left(\frac{\sin\pi u}{\pi u}\right)^2\frac{1}{1-u^2}$ which is the Selberg minorant of the characteristic function of the interval $[-1,1]$ with Fourier transform $\hat{h}(\alpha) = \max(1-|\alpha|+\frac{\sin 2\pi|\alpha|}{2\pi},0)$ was taken (the use of the latter Fourier pair in such a context goes back to [7]). Let m_{t_1} denote the multiplicity of a zero $\frac{1}{2}+t_1$ of $\zeta\cdot Z_1(s)$, and $\lambda\in(0,1)$. Then we see that

$$\sum_{0<t_1\le T} m_{t_1} + 2\sum_{\substack{0<t_1,t_2\le T\\ 0<t_1-t_2\le\frac{\pi\lambda}{\log T}}} 1 \ge \sum_{0<t_1,t_2\le T}\frac{4}{4+(t_1-t_2)^2}\,h\left((t_1-t_2)\frac{\log T}{\pi\lambda}\right), \tag{199}$$

and since

$$h\left(\frac{2u}{\lambda}\right) = \int_{-\infty}^{\infty} \hat{h}(\alpha) e^{2\pi i \alpha \frac{2u}{\lambda}}\, d\alpha = \frac{\lambda}{2} \int_{-\infty}^{\infty} \hat{h}\left(\frac{\lambda\beta}{2}\right) e^{2\pi i \beta u}\, d\beta,$$

we have

$$\sum_{0<t_1\le T} m_{t_1} + 2 \sum_{\substack{0<t_1,t_2\le T \\ 0<t_1-t_2\le \frac{\pi\lambda}{\log T}}} 1 \gtrsim \left(\frac{T}{\pi}\log T\right) \frac{\lambda}{2} \int_{-\infty}^{\infty} \mathscr{F}(\beta)\hat{h}\left(\frac{\lambda\beta}{2}\right) d\beta$$

$$\gtrsim \left(\frac{T}{\pi}\log T\right) \frac{\lambda}{2} \int_{-1+\epsilon}^{1-\epsilon} \mathscr{F}(\beta)\hat{h}\left(\frac{\lambda\beta}{2}\right) d\beta.$$

In the last step we have used the non-negativity of $\hat{h}$ and $\mathscr{F}$, the latter seen in the same way as (3). We may assume that

$$\sum_{0<t_1\le T} m_{t_1} \sim \frac{T}{\pi}\log T,$$

for otherwise there would be infinitely many multiple zeros of $\zeta \cdot Z_1$ and there wouldn't be any need to prove a result for existence of small gaps between zeros. Then

$$\sum_{\substack{0<t_1,t_2\le T \\ 0<t_1-t_2\le \frac{\pi\lambda}{\log T}}} 1 \ge \left(\frac{1}{2}-\epsilon\right)\left(\frac{T}{\pi}\log T\right)\left(-1+\frac{\lambda}{2}\int_{-1+\epsilon}^{1-\epsilon} \mathscr{F}(\beta)\hat{h}\left(\frac{\lambda\beta}{2}\right) d\beta\right). \tag{200}$$

Since this holds for any fixed $\epsilon > 0$, we want to find an as small as possible value of λ for which

$$-1 + \frac{\lambda}{2}\int_{-1+\epsilon}^{1-\epsilon} \mathscr{F}(\beta)\hat{h}\left(\frac{\lambda\beta}{2}\right) d\beta > 0. \tag{201}$$

Choosing the Fourier pair h, $\hat{h}$ as in [5], by (198) the condition (201) is cast into

$$\lambda \int_0^1 \left[2\beta - 4\beta^2 + \frac{1}{2}\sum_{m=1}^{\infty} \frac{(m-1)!}{(2m)!}(2\beta)^{2m+1} + 2T^{-2\beta}\log T\right]\left(1 - \frac{\lambda\beta}{2} + \frac{\sin\pi\lambda\beta}{2\pi}\right) d\beta$$

$$> 1. \tag{202}$$

Some computing reveals that (202) is satisfied when we take $\lambda = 0.78842$, and we can state

Corollary 2 *Assume RH. There exist infinitely many gaps of size* < 0.78842 *of the average gap* ($\sim \dfrac{\pi}{\log T}$) *in the sequence composed of the zeros and maxima points on the critical line of* $\zeta(s)$.

Remark Since the critical zeros of ζ and Z_1 are interlaced these small gaps are between a zero of ζ and a zero of Z_1. The result of Corollary 2 doesn't seem not very strong, but it is better than the trivial bound 1, and it is the first result of its kind. Applying (198) for the proportion of simple zeros of $\zeta \cdot Z_1$ doesn't give a bright result. By some elementary considerations and Hall's result given in (55) we may take that a non-simple zero $\frac{1}{2} + it_1$ of $\zeta \cdot Z_1$ has multiplicity at least 3. Then, Montgomery's argument for the proportion of simple zeros, together with $\sum_{\substack{0<t_1\le T \\ t_1:\,\text{simple}}} 1 \ge \sum_{0<t_1\le T} \dfrac{3-m_{t_1}}{2}$ and a numerical fact from the proof of Corollary 1.3 of [5], gives the rather dismal estimate $\sum_{\substack{0<t_1\le T \\ t_1:\,\text{simple}}} 1 \gtrsim 0.38\left(\dfrac{T}{\pi}\log T\right)$. That these results aren't as strong as those obtained from $F_{\zeta,\zeta}(x,T)$ by Montgomery is related to the fact that $F_{\zeta,Z_1}(x,T)$ changes sign at $x \approx \sqrt{T}$.

9 Correlation of the Zeros of Two Dirichlet *L*-Functions

In this section, we apply the ideas of Sects. 3 and 4 to correlating the zeros of two Dirichlet L-functions associated with primitive Dirichlet characters. Let χ be a primitive character to the moduli q_χ (so that $q_\chi \ge 3$). In this section the Dirichlet characters will always be taken to be primitive characters to moduli which are fixed (i.e. not a function of T). The non-trivial zeros of $L(s,\chi)$ have real parts in the strip $\sigma \in (0,1)$, and the number of these with ordinates in $[0,T]$ is

$$N_\chi(T) = \frac{T}{2\pi}\log\frac{q_\chi T}{2\pi} - \frac{T}{2\pi} + O(\log q_\chi T), \quad (T \ge 4) \tag{203}$$

(see, e.g., Corollary 14.7 of [19]). We shall assume the Generalized Riemann Hypothesis (GRH) for Dirichlet L-functions that all of the non-trivial zeros of $L(s,\chi)$ are on the critical line. Let $\rho_\chi = \frac{1}{2} + i\gamma_\chi$ run through these zeros. Quoting from [23], the analogue of Montgomery's explicit formula (19) is

$$\sum_{\gamma_\chi} \frac{(2\sigma-1)x^{i\gamma_\chi}}{(\sigma-\frac{1}{2})^2 + (t-\gamma_\chi)^2} = -x^{-\frac{1}{2}}\left(\sum_{n\le x}\Lambda(n)\chi(n)\left(\frac{x}{n}\right)^{1-\sigma+it} + \sum_{n>x}\Lambda(n)\chi(n)\left(\frac{x}{n}\right)^{\sigma+it}\right)$$
$$-\frac{L'}{L}(1-\sigma+it,\chi)x^{\frac{1}{2}-\sigma+it}$$

$$+x^{-\frac{1}{2}}\sum_{r=1}^{\infty}\frac{(2\sigma-1)x^{-2r-\mathfrak{a}_\chi}}{(\sigma-1-it-2r-\mathfrak{a}_\chi)(\sigma+it+2r+\mathfrak{a}_\chi)}, \tag{204}$$

valid under GRH for $\sigma > 1$, and all $x \geq 1$. Here

$$\mathfrak{a}_\chi = \begin{cases} 0 & \text{if} \quad \chi(-1) = 1, \\ 1 & \text{if} \quad \chi(-1) = -1. \end{cases}$$

We employ (204) with $\sigma = \frac{5}{2}$, then while going through a procedure paralleling that of Sect. 3 we substitute

$$\frac{L'}{L}(1-\sigma+it,\chi) = -\log\left(\frac{q_\chi|t|}{2\pi}\right) - \frac{L'}{L}(\sigma-it,\overline{\chi}) + O\left(\frac{1}{|t|}\right), \ (-2<\sigma<3,\ |t|\geq 1) \tag{205}$$

which follows from logarithmic differentiation of the functional equation of $L(s,\chi)$, and we obtain the analogue of (22)

$$\begin{aligned}\sum_{\gamma_\chi}\frac{4x^{i(\gamma_\chi-t)}}{4+(\gamma_\chi-t)^2} &= -x^{-2}\sum_{n\leq x}\Lambda(n)\chi(n)n^{\frac{3}{2}-it} - x^2\sum_{n>x}\Lambda(n)\chi(n)n^{-\frac{5}{2}-it} \\ &\quad + x^{-2}\big(\log q_\chi|t| + O(1)\big)\end{aligned} \tag{206}$$

for $|t| \geq 1$ and $x \geq 1$.

Now let ψ be a primitive character to the modulus q_ψ, and let t run through the ordinates $\gamma_\psi \in (1,T]$ of the zeros of $L(s,\psi)$. Summing (206) over this set of t values, we have

$$\begin{aligned}\sum_{\gamma_\chi}\sum_{1<\gamma_\psi\leq T}\frac{4x^{i(\gamma_\chi-\gamma_\psi)}}{4+(\gamma_\chi-\gamma_\psi)^2} &= \frac{T\log^2 T}{2\pi x^2}\left(1+O_{q_\chi,q_\psi}\left(\frac{1}{\log T}\right)\right) \\ &\quad - x^{-2}\sum_{n\leq x}\Lambda(n)\chi(n)n^2\sum_{1<\gamma_\psi\leq T}n^{-\rho_\psi} \\ &\quad - x^2\sum_{n>x}\Lambda(n)\chi(n)n^{-2}\sum_{1<\gamma_\psi\leq T}n^{-\rho_\psi},\end{aligned} \tag{207}$$

where we have used

$$\sum_{1<\gamma_\psi\leq T}\big(\log q_\chi\gamma_\psi + O(1)\big) = \frac{T}{2\pi}\log^2 T\left(1+O_{q_\chi,q_\psi}\left(\frac{1}{\log T}\right)\right) \tag{208}$$

which follows from (203) by Stieltjes integration.

At this point we need the analogue of the Landau–Gonek formula (25) for $L(s,\psi)$ to be used in (207). A meticulous GRH dependent version was given by Fujii [6].

Proposition 6 *For $y, T > 1$ we have*

$$\sum_{1<\gamma_\psi\le T} y^{\rho_\psi} = -\frac{T}{2\pi}\psi(y)\Lambda(y) + O(y\log(yq_\psi T)\log\log 3y) + O\left(\log y \min\left(T, \frac{y}{\langle y\rangle}\right)\right)$$
$$+ O\left(\log(q_\psi T)\min\left(T, \frac{1}{\log y}\right)\right) + O\left(\frac{y\log^2 q_\psi}{\log 2y}\right). \tag{209}$$

The proof of this proposition is similar to Gonek's proof of (25) in [13], so we omit it. The only slight difference is in estimating the contribution of the segment along $t = 1$ (in case there is no $\gamma_\psi = 1$; if there is a $\gamma_\psi = 1$, then this segment can be shifted a little bit), we use $\frac{L'}{L}(\sigma + i) \ll \log^2 q_\psi$ for $-1 < \sigma < 2$, giving rise to the last error term in (209). If GRH is assumed, as we have already done, and if $y = n \ge 2$, then the argument gives

$$\sum_{1<\gamma_\psi\le T} n^{-\rho_\psi} = -\frac{T}{2\pi n}\overline{\psi(n)}\Lambda(n) + O(\log n\log\log n) + O(\log q_\psi T) + O\left(\frac{\log^2 q_\psi}{\log n}\right). \tag{210}$$

Plugging (210) into (207) we have

$$\sum_{\gamma_\chi}\sum_{1<\gamma_\psi\le T}\frac{4x^{i(\gamma_\chi-\gamma_\psi)}}{4+(\gamma_\psi-\gamma_\chi)^2} = \frac{T\log^2 T}{2\pi x^2}\left(1 + O_{q_\chi,q_\psi}\left(\frac{1}{\log T}\right)\right) \tag{211}$$
$$+ \frac{T}{2\pi x^2}\sum_{n\le x} n\Lambda^2(n)(\overline{\psi}\chi)(n) + \frac{Tx^2}{2\pi}\sum_{n>x}\frac{\Lambda^2(n)(\overline{\psi}\chi)(n)}{n^3}$$
$$+ O(x\log 2x\log\log 3x) + O(x\log q_\psi T) + O\left(\frac{x\log^2 q_\psi}{\log 2x}\right).$$

The summation conditions for the double sum on the left-hand side can be changed into $0 \le \gamma_\chi, \gamma_\psi \le T$ within an error of $O(\log(q_\chi T)\log(q_\psi T)\log T)$ as can be seen by a standard estimation.

If $\chi \ne \psi$, then from

$$\sum_{n\le x}\Lambda(n)(\overline{\psi}\chi)(n) \ll x^{\frac{1}{2}}\log^2 x, \quad (q_\chi q_\psi \le x), \tag{212}$$

on GRH [2, Chap. 20], we have

$$\frac{T}{2\pi x^2}\sum_{n\le x} n\Lambda^2(n)(\overline{\psi}\chi)(n) + \frac{Tx^2}{2\pi}\sum_{n>x}\frac{\Lambda^2(n)(\overline{\psi}\chi)(n)}{n^3} \ll Tx^{-\frac{1}{2}}\log^3 2x, \tag{213}$$

so that there is no main term of size $T\log T$ for $x > T^\epsilon$.

If $\chi = \psi$, then with $q_\chi = q_\psi \le x$, the terms in (213) are the same as in (26) for the pair correlation of zeros of $\zeta(s)$ except for the summands with $(n, q_\chi) > 1$. The effect of these latter terms are

$$\ll \frac{T}{x^2} \sum_{\substack{n \le x \\ (n,q_\chi)>1}} n\Lambda^2(n) \ll \frac{T \log x \log q_\chi}{x}. \tag{214}$$

This completes our calculation in this section, and we state our findings as

Theorem 3 *Assume GRH. Let $\frac{1}{2} + i\gamma_\chi$ and $\frac{1}{2} + i\gamma_\psi$ run through the critical zeros of $L(s, \chi)$ and $L(s, \psi)$, respectively. Assume that the moduli q_χ, q_ψ are fixed. Then, as $T \to \infty$,*

$$\begin{aligned} F_{\chi,\psi}(x, T) &:= \sum_{0<\gamma_\chi \le T} \sum_{0<\gamma_\psi \le T} \frac{4x^{i(\gamma_\chi - \gamma_\psi)}}{4 + (\gamma_\chi - \gamma_\psi)^2} \\ &= [\chi = \psi] \cdot \left(\frac{T \log x}{2\pi} + O\left(\frac{T \log x \log q_\chi}{x} \right) \right) \\ &\quad + \frac{T \log^2 T}{2\pi x^2} \left(1 + O_{q_\chi, q_\psi}\left(\frac{1}{\log T} \right) \right) \\ &\quad + O(Tx^{-\frac{1}{2}} \log^3 2x) + O(x \log 2x \log\log 3x) + O(x \log T). \end{aligned} \tag{215}$$

Remark We note that if $\psi \equiv 1$ is taken then the preceding calculation is the same, and so we obtain the correlation of critical zeros of $\zeta(s)$ with those of $L(s, \chi)$.

Remark If χ is a non-real character and $\psi = \overline{\chi}$, then the zeros of $L(s, \psi)$ occurring in $F_{\chi,\psi}(x, T)$ are the reflections with respect to the real axis of the zeros of $L(s, \chi)$ which are below the real axis, and for $x \ge T^\epsilon$ the main term is again 0. Thus the zeros of $L(s, \chi)$ which are above the real axis can be said to be not correlated to those which are below the real axis.

Remark In [23] a certain average over correlations of zeros of two Dirichlet L-functions of the same modulus were obtained by following Montgomery's method. Our theorem here is more particular.

Remark Gallagher's citation of oral communication in [7] seems to indicate that a result of the same kind as Theorem 3 was observed by A. Fujii.

We now work on the product $L(s, \chi) \cdot L(s, \psi)$, where $\chi \ne \psi$ are primitive characters. Similar to (196), let

$$\mathfrak{F}(\alpha) := \left(\frac{T}{\pi} \log T \right)^{-1} \left[F_{\chi,\chi}(T^\alpha, T) + 2F_{\chi,\psi}(T^\alpha, T) + F_{\psi,\psi}(T^\alpha, T) \right], \tag{216}$$

so that

$$\mathfrak{F}(\alpha) = \left(\frac{T}{\pi}\log T\right)^{-1} \sum_{\substack{0<t_1,t_2\le T \\ L(\frac{1}{2}+it_j,\chi)\cdot L(\frac{1}{2}+it_j,\psi)=0,\,(j=1,2)}} \frac{4}{4+(t_1-t_2)^2}\, T^{i\alpha(t_1-t_2)}. \tag{217}$$

Then we know from Theorem 3 that, for $0 \le |\alpha| \le 1-\epsilon$ with any arbitrarily small but fixed ϵ,

$$\mathfrak{F}(\alpha) \sim |\alpha| + 2T^{-2|\alpha|}\log T, \quad \text{as } T \to \infty. \tag{218}$$

For small gaps between zeros of $L(s,\chi)\cdot L(s,\psi)$, proceeding as in (199)–(202), we reach the condition

$$\lambda \int_0^1 \left[\beta + 2T^{-2\beta}\log T\right]\left(1 - \frac{\lambda\beta}{2} + \frac{\sin \pi\lambda\beta}{2\pi}\right) d\beta > 1. \tag{219}$$

Some computing reveals that (219) is satisfied with $\lambda = 0.68742$, and we have

Corollary 3 *Assume RH. There exist infinitely many gaps of size* < 0.68742 *of the average gap* ($\sim \dfrac{\pi}{\log T}$) *in the sequence composed of the zeros of* $L(s,\chi)\cdot L(s,\psi)$, *where* $\chi \ne \psi$ *are primitive characters.*

Note that the small gaps detected here may be coming from a single L-function.

10 Further Problems to Study

The method presented in this article may be applied to correlating the ordinates of zeros of the Riemann zeta-function or Dirichlet L-functions to various sequences of real numbers (and possibly generalizable to other functions from the Selberg class). Moreover as in Sect. 8 one may try to work out the pair correlation of zeros of other related functions such as $Z_1(s)$. Furthermore, our method is applicable to products of not just two Dirichlet L-functions, but in fact to products of k Dirichlet L-functions. In any such problem it is desirable to obtain results which have implications for the distribution of zeros on the critical line. Another problem of interest is to form conjectures about the behaviour of the correlations of Theorems 1 and 2 for $x \ge T$ as Montgomery [17] did for the pair correlation of zeta zeros. Montgomery drew upon the Hardy–Littlewood conjecture about sums of the kind $\sum_{n\le x}\Lambda(n)\Lambda(n+h)$ for formulating the conjecture given in (8). The conjecture that all zeta zeros are simple also provided guidance in the sense that his conjecture should agree with it. In the case of the correlation sums of Theorems 1 and 2, extension to $x \ge T$

will involve formulating Hardy–Littlewood type of conjectures for the arithmetic functions $\Lambda^k(n)$ and $\lambda^k(n)$, or else one may rely upon the random matrix model to first formulate the analogues of the pair correlation conjecture (10) and then pass to the analogues of (8) therefrom.

Acknowledgements C.Y. Yıldırım thanks Professors Enrico Bombieri and Andrew Granville for helpful conversations and suggestions.

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Bagchi's Theorem for Families of Automorphic Forms

E. Kowalski

Abstract We prove a version of Bagchi's Theorem and of Voronin's Universality Theorem for the family of primitive cusp forms of weight 2 and prime level, and discuss under which conditions the argument will apply to a general reasonable family of automorphic L-functions.

1 Introduction

The first "universality theorem" for Dirichlet series is Voronin's Theorem [18] for the Riemann zeta function, which states that for any $r < 1/4$, any continuous function φ defined and non-vanishing on the disc $|s - 3/4| \leqslant r$ in $\mathbf{C}$, which is holomorphic in the interior, and any $\varepsilon > 0$, there exists $t \in \mathbf{R}$ such that

$$\max_{|s-3/4|\leqslant r} |\zeta(s + it) - \varphi(s)| < \varepsilon.$$

In other words, up to arbitrary precision, any function φ can be approximated by some vertical translate of the Riemann zeta function.

Bagchi, in his thesis [1], provided a clear conceptual explanation of this result, as the combination of two independent statements:

- Viewing translates of the Riemann zeta function by $t \in [-T, T]$ as random variables with values in a space of holomorphic function on the disc, Bagchi proves that these random variables converge in law, as $T \to +\infty$, to a natural random Dirichlet series, which is also expressed as a random Euler product;
- Computing the support of the limiting random Dirichlet series, and checking that it contains the space of nowhere vanishing holomorphic functions on the disc, the universality theorem follows easily.

The key step, from our point of view, is the first part, which we call *Bagchi's Theorem*. Indeed, once the convergence in law is known, it follows that there is

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H. Montgomery et al. (eds.), *Exploring the Riemann Zeta Function*,
DOI 10.1007/978-3-319-59969-4_8

"some" universality statement, with respect to the functions in the support of the limiting random Dirichlet series. The second step makes this support explicit. (This might be compared with Deligne's Equidistribution Theorem, as applied to families of exponential sums, for instance: Deligne's Theorem shows that there is always *some* equidistribution of these sums.)

The goal of this note is to give a first example of a genuinely higher-degree statement of this type, and to deduce the corresponding universality statement. We will also indicate a general principle that should apply in many more cases.

Theorem 1 (Universality in Level Aspect) *For q prime $\geqslant 17$, let $S_2(q)^*$ be the non-empty*[1] *finite set of primitive cusp forms for $\Gamma_0(q)$ with weight 2 and trivial nebentypus. For $f \in S_2(q)^*$, let $L(f,s)$ denote its Hecke L-function*

$$L(f,s) = \sum_{n \geqslant 1} \lambda_f(n) n^{-s},$$

normalized so that the critical line is $\mathrm{Re}(s) = \frac{1}{2}$.

For any real number $r < \frac{1}{4}$, let D be the open disc centered at $3/4$ with radius r. Then for any continuous function $\varphi : \bar{D} \to \mathbf{C}$ which is holomorphic and non-vanishing in D and satisfies

$$\varphi(\sigma) > 0 \text{ for } \sigma \in D \cap \mathbf{R}, \tag{1}$$

we have

$$\liminf_{q \to +\infty} \frac{1}{|S_2(q)^*|} |\{f \in S_2(q)^* \mid \|L(f,\cdot) - \varphi\|_\infty < \varepsilon\}| > 0$$

for any $\varepsilon > 0$, where the L^∞ norm is the norm on $\bar{D}$.

The main difference with previous results involving cusp forms (the first one being due to Laurinčikas and Matsumoto [13]) is that we do not fix one such L-function $L(f,s)$ and consider shifts (or twists) $L(f, s+it)$ or $L(f \times \chi, s)$, but rather we average over the discrete family of primitive forms in $S_2(q)^*$. It is also important to remark that the condition (1) is necessary for a function on D to be approximated by L-functions $L(f,s)$ with $f \in S_2(q)^*$. (We will give more general statements where the discs D are replaced with more general compact sets in the strip $\frac{1}{2} < \sigma < 1$).

We will prove this Theorem in Sect. 2, after stating the results generalizing the two steps of Bagchi's strategy for the zeta function. The proof of Bagchi's Theorem for this family is an analogue of a proof for the Riemann zeta function that is simpler than Bagchi's proof (it avoids both the use of the ergodic theorem and any tightness or weak-compactness argument).

[1] We assume $q \geqslant 17$ to ensure this property; it also holds for $q = 11$.

In Sect. 5, we discuss very briefly how this strategy can in principle be applied to very general families of L-functions, as defined in [8].

Acknowledgements

The simple proof of Bagchi's Theorem for the Riemann zeta function, that we generalize here to modular forms, was found during a course on probabilistic number theory at ETH Zürich in the Fall Semester 2015; thanks are due to the students who attended this course for their interest and remarks, and to B. Löffel for assisting with the exercises (see [9] for a draft of the lecture notes for this course, especially Chapter 3).

Thanks to the referee for carefully reading the text, and in particular for pointing out a number of confusing mistakes in references to the literature.

Notation

As usual, $|X|$ denotes the cardinality of a set. By $f \ll g$ for $x \in X$, or $f = O(g)$ for $x \in X$, where X is an arbitrary set on which f is defined, we mean synonymously that there exists a constant $C \geqslant 0$ such that $|f(x)| \leqslant Cg(x)$ for all $x \in X$. The "implied constant" is any admissible value of C. It may depend on the set X which is always specified or clear in context. We write $f \asymp g$ if $f \ll g$ and $g \ll f$ are both true.

We use standard probabilistic terminology: a probability space $(\Omega, \Sigma, \mathbf{P})$ is a triple made of a set Ω with a σ-algebra and a measure $\mathbf{P}$ on Σ with $\mathbf{P}(\Omega) = 1$. We denote by $\mathbf{E}(X)$ the expectation on Ω. The law of a random variable X is the measure ν on the target space of X defined by $\nu(A) = \mathbf{P}(X \in A)$. If $A \subset \Omega$, then $\mathbf{1}_A$ is the characteristic function of A.

2 Equidistribution and Universality for Modular Forms in the Level Aspect

We will prove Theorem 1 by combining the results of the following two steps, each of which will be proved in a forthcoming section. Throughout, we assume that q is a prime number $\geqslant 17$.

Step 1. (Equidistribution; Bagchi's Theorem) For q prime, we view the finite set $S_2(q)^*$ as a probability space with the probability measure proportional to the "harmonic" measure where $f \in S_2(q)^*$ has weight

$$\frac{1}{\langle f, f\rangle}$$

in terms of the Petersson inner product. We write correspondingly $\mathbf{E}_q(\cdot)$ or $\mathbf{P}_q(\cdot)$ for the corresponding expectation and probability. Hence there exists a constant $c_q > 0$ such that

$$\mathbf{E}_q(\varphi(f)) = \sum_{f \in S_2(q)^*} \frac{c_q}{\langle f, f\rangle} \varphi(f)$$

for any $\varphi : S_2(q)^* \to \mathbf{C}$. From the Petersson formula, it is known that $c_q \to 1/(4\pi)$ as $q \to +\infty$ (see, e.g., Iwaniec and Kowalski [7, Chap. 14] or Cogdell and Michel [3]).

Let D be a relatively compact open set in $\mathbf{C}$ such that D is invariant under complex conjugation. Define $\mathcal{H}(D)$ to be the Banach space of functions holomorphic on D and continuous and bounded on $\bar{D}$, with the norm

$$\|\varphi\|_\infty = \sup_{s \in \bar{D}} |\varphi(s)|.$$

This is a separable complex Banach space. Define also $\mathcal{H}_{\mathbf{R}}(D)$ to be the set of $\varphi \in \mathcal{H}(D)$ such that $f(\bar{s}) = \overline{f(s)}$ for all $s \in D$ (this is well-defined since C is assumed to be invariant under conjugation). Note that the L-function of f, restricted to D, is an element of $\mathcal{H}_{\mathbf{R}}(D)$ since the Hecke eigenvalues $\lambda_f(n)$ are real for all $n \geqslant 1$.

We define L_q to be the random variable $S_2(q)^* \to \mathcal{H}(D)$ mapping $f \in S_2(q)^*$ to the restriction of $L(f, s)$ to D. (This depends on D, but the choice of D will always be clear in the context.)

If $\bar{D}$ is a compact subset of the strip $\frac{1}{2} < \mathrm{Re}(s) < 1$, then we will show that L_q converges in law to a random Dirichlet series. To define the limit, let $(X_p)_p$ be a sequence of independent random variables indexed by primes, taking values in the matrix group $\mathrm{SU}_2(\mathbf{C})$ and distributed according to the probability Haar measure on $\mathrm{SU}_2(\mathbf{C})$.

Theorem 2 (Bagchi's Theorem for Modular Forms) *Assume that $\bar{D}$ is a compact subset of the strip $\frac{1}{2} < \mathrm{Re}(s) < 1$. Then, as $q \to +\infty$, the random variables L_q converge in law to the random Euler product*

$$L_D(s) = \prod_p \det(1 - X_p p^{-s})^{-1} = \prod_p (1 - \mathrm{Tr}(X_p) p^{-s} + p^{-2s})^{-1}$$

which is almost surely convergent in $\mathcal{H}(D)$, and belongs almost surely to $\mathcal{H}_{\mathbf{R}}(D)$.

Step 2. (Support of the random Euler product)

To deduce Theorem 1 from Theorem 2, we need the following computation of the support of the limiting measure.

Theorem 3 *Suppose that D is a disc with positive radius and diameter a segment of the real axis, always with $\bar{D}$ contained in $\frac{1}{2} < \mathrm{Re}(s) < 1$. The support of the law of the random Euler product L_D contains the set of functions $\varphi \in \mathcal{H}(D)$ such that $\varphi(x) > 0$ for $x \in D \cap \mathbf{R}$.*

Note that since $D \cap \mathbf{R}$ is an interval of positive length in $\mathbf{R}$, the condition $\varphi(x) > 0$ for all $x \in D \cap \mathbf{R}$ implies by analytic continuation that $\varphi \in \mathcal{H}_{\mathbf{R}}(D)$, which by Bagchi's Theorem 2 is a necessary condition to be in the support of L_D.

Step 3. (Conclusion) The elementary Lemma 1 below, combined with Theorems 2 and 3, implies Theorem 1 in the form

$$\liminf_{q\to+\infty} \mathbf{P}_q(\|L(f,\cdot)-\varphi\|_\infty < \varepsilon) = \liminf_{q\to+\infty} \sum_{\substack{f\in S_2(q)^* \\ \|L(f,\cdot)-\varphi\|_\infty<\varepsilon}} \frac{c_q}{\langle f,f\rangle} > 0 \tag{2}$$

for any function φ as in Theorem 1 and any $\varepsilon > 0$. We can easily deduce the "natural density" version from this: let A be the set of those $f \in S_2(q)^*$ such that $\|L(f,\cdot)-\varphi\|_\infty < \varepsilon$; then for any parameter $\eta > 0$, the definition of the harmonic measure on $S_2(q)^*$ gives

$$\frac{1}{|S_2(q)^*|}\sum_{f\in A} 1 = \mathbf{E}_q\Big(\mathbf{1}_A(f)\frac{\langle f,f\rangle}{c_q|S_2(q)^*|}\Big) \geqslant \eta\Big(\mathbf{P}_q(A) - \mathbf{P}_q\Big(\frac{\langle f,f\rangle}{c_q|S_2(q)|^*} < \eta\Big)\Big).$$

There exists $\delta > 0$ such that the first term is $\geqslant \delta > 0$ for all q large enough by (2); on the other hand, a result of Cogdell and Michel [3, Corollary 1.16] and the classical relation between the Petersson norm and the symmetric square L-function at $s = 1$ (see, e.g., [7, (5.101)]) imply that we can find $\eta > 0$ such that

$$\lim_{q\to+\infty} \mathbf{P}_q\Big(\frac{\langle f,f\rangle}{c_q|S_2(q)|^*} < \eta\Big) < \frac{\delta}{2}.$$

For this value of η, we obtain

$$\liminf_{q\to+\infty} \frac{1}{|S_2(q)^*|}\sum_{f\in A} 1 \geqslant \frac{\eta\delta}{2} > 0.$$

More precisely, the result of Cogdell–Michel is that for any $\eta > 0$, we have

$$\lim_{q\to+\infty} \mathbf{P}_q(L(\mathrm{Sym}^2 f, 1) \leqslant \eta) = F(\log\eta)$$

where F is the limiting distribution function for the special value at 1 of the symmetric square L-function of $f \in S_2(q)^*$. Since $F(x) \to 0$ when $x \to -\infty$, we obtain the result.

Remark 1 It would also be possible to argue throughout with the uniform probability measure on $S_2(q)^*$; the only change would be a slightly different form of Theorem 2, where the random variables (X_p) would not be identically distributed (compare with the equidistribution theorems of Serre [16] and Conrey et al. [4]).

Lemma 1 *Let M be a separable complete metric space and (X_n) a sequence of random variables with values in M that converge in law to X. Let S be the support of the law of X. Then for any $x \in S$, and any open neighborhood U of x, we have*

$$\liminf_{n\to+\infty} \mathbf{P}(X_n \in U) > 0.$$

Proof By classical criteria for convergence in law, we have

$$\liminf_{n\to+\infty} \mathbf{P}(X_n \in U) \geqslant \mathbf{P}(X \in U) \tag{3}$$

for any open set $U \subset X$ (see, e.g., [2, Theorem 2.1 (iv)]). Since $x \in S$, we have $\mathbf{P}(X \in U) > 0$, hence the result. □

3 Proof of Theorem 2

We begin with some preliminaries concerning the random Euler product L_D. In fact, if will be convenient to view it as a holomorphic function on larger domains then D, in a way that will be clear below. For this purpose, we fix a real number σ_0 such that $\frac{1}{2} < \sigma_0 < 1$, and such that the compact set $\bar{D}$ is contained in the half-plane S defined by $\mathrm{Re}(s) > \sigma_0$.

We recall that for $\nu \geqslant 0$, the d-th Chebyshev polynomial is defined by

$$U_\nu(2\cos(x)) = \frac{\sin((\nu+1)x)}{\sin(x)}.$$

The importance of these polynomials for us lies in their relation with the representation theory of $\mathrm{SU}_2(\mathbf{C})$, namely

$$U_\nu(2\cos(x)) = \mathrm{Tr}\left(\mathrm{Sym}\begin{pmatrix} e^{ix} & 0 \\ 0 & e^{-ix}\end{pmatrix}\right)$$

for any $x \in \mathbf{R}$, where Sym^d is the d-th symmetric power of the standard two-dimensional representation of $\mathrm{SU}_2(\mathbf{C})$.

We define a sequence of random variables $(Y_n)_{n\geqslant 1}$ by

$$Y_n = \prod_{p^\nu || n} U_\nu(\mathrm{Tr}(X_p)).$$

In particular, we have $Y_n Y_m = Y_{nm}$ if n and m are coprime, and $Y_p = \mathrm{Tr}(X_p)$ if p is prime. The sequence (Y_p) is independent and Sato–Tate distributed. Moreover, since $|U_\nu(t)| \leqslant \nu + 1$ for all $\nu \geqslant 0$ and all $t \in \mathbf{R}$, we have

$$|Y_n| \leqslant \prod_{p^\nu || n} (\nu+1) = d(n) \ll n^\varepsilon$$

for $n \geqslant 1$ and $\varepsilon > 0$, where the implied constant depends only on ε.

Lemma 2 (1) *Almost surely, the random Euler product*

$$\prod_p \det(1 - X_p p^{-s})^{-1}$$

converges and defines a holomorphic function on S. In particular, L_D converges almost surely to define an $\mathcal{H}(D)$-valued random variable.

(2) *Almost surely, we have*

$$\prod_p \det(1 - X_p p^{-s})^{-1} = \sum_{n \geqslant 1} Y_n n^{-s}$$

for all $s \in S$, and in particular L_D coincides with the random Dirichlet series on the right.

(3) *For $\sigma > 1/2$ and $u \geqslant 2$, we have*

$$\mathbf{E}\Big(\Big|\sum_{n \leqslant u} Y_n n^{-\sigma}\Big|^2\Big) \ll 1,$$

where the implied constant depends only on σ.

Proof (1) Let σ be a fixed real number such that $\frac{1}{2} < \sigma < \sigma_0$. By expanding, we can write

$$-\log \det(1 - X_p p^{-s}) = Y_p p^{-s} + g_p(s)$$

where the random series

$$\sum_p g_p(s)$$

converges absolutely (and surely) for $\mathrm{Re}(s) > 1/2$. Since $\mathbf{E}(Y_p p^{-\sigma}) = 0$ and $\mathbf{E}(Y_p^2 p^{-2\sigma}) = p^{-2\sigma}$, Kolmogorov's Three Series Theorem (see, e.g., [14, Theorem 0.III.2]) implies that the random series

$$\sum_p Y_p p^{-\sigma}$$

converges almost surely. By well-known results on Dirichlet series, this means that the random series

$$\sum_p Y_p p^{-s}$$

converges almost surely to a holomorphic function on the half-plane S. This implies the first statement by taking the exponential. The second follows by restricting to D since $\bar{D}$ is contained in S.

(2) We first show that, almost surely, the random Dirichlet series

$$\tilde{L}(s) = \sum_{n\geqslant 1} Y_n n^{-s}$$

converges and defines a function holomorphic on S. The key point is that the variables Y_n for n squarefree form an orthonormal system: we have

$$\mathbf{E}(Y_n Y_m) = \delta(n, m)$$

if n and m are squarefree numbers. Indeed, if $n \neq m$, there is a prime p dividing only one of n and m, say $p \mid n$, and then by independence we get $\mathbf{E}(Y_n Y_m) = \mathbf{E}(Y_p)\mathbf{E}(Y_{n/p}Y_m) = 0$; and if $n = m$ is squarefree, then we have

$$\mathbf{E}(Y_n^2) = \prod_{p|n} \mathbf{E}(Y_p^2) = 1.$$

Fix again σ such that $\frac{1}{2} < \sigma < \sigma_0$. By the Rademacher–Menchov Theorem (see, e.g., [9, Theorem B.8.4]), the random series

$$\sum_n^{\flat} Y_n n^{-\sigma}$$

over squarefree numbers converges almost surely. By elementary factorization and properties of products of Dirichlet series (the product of an absolutely convergent Dirichlet series and a convergent one is convergent, see, e.g., [6, Theorem 54]) the same holds for

$$\sum_{n\geqslant 1} Y_n n^{-\sigma}.$$

As in (1), this gives the almost sure convergence of the series defining $\tilde{L}(s)$ to a holomorphic function in S. Restricting gives the $\mathcal{H}(D)$-valued random variable $\tilde{L}_D$.

Finally, almost surely both the random Euler product and the random Dirichlet series converge and are holomorphic for $\mathrm{Re}(s) > \sigma_0$. For $\mathrm{Re}(s) > 3/2$, they converge absolutely, and coincide by a well-known formal Euler product computation: for any prime p and any $x \in \mathbf{R}$, denoting

$$t(x) = \begin{pmatrix} e^{ix} & 0 \\ 0 & e^{-ix} \end{pmatrix}$$

we have

$$\det(1-t(x)p^{-s})^{-1} = (1-e^{ix}p^{-s})^{-1}(1-e^{-ix}p^{-s})^{-1}$$
$$= \sum_{\nu\geqslant 0} \operatorname{Tr}\operatorname{Sym}^{\nu}(t(x))p^{-\nu s} = \sum_{\nu\geqslant 0} U_{\nu}(x)p^{-\nu s}$$

(compare the discussion of Cogdell and Michel in [3, Sect. 2]). By analytic continuation, we deduce that $L_D = \tilde{L}_D$ almost surely as $\mathcal{H}(D)$-valued random variables.

(3) Since the random variables Y_n are real-valued, we have

$$\mathbf{E}\Big(\Big|\sum_{n\leqslant u} Y_n n^{-\sigma}\Big|^2\Big) = \sum_{n,m\leqslant u} \frac{1}{(nm)^{\sigma}}\mathbf{E}(Y_nY_m).$$

For given n and m, let $d=(n,m)$ and $n'=n/d$, $m'=m/d$. Then by multiplicativity and independence of the variables $(Y_p)_p$, we have

$$\mathbf{E}(Y_nY_m) = \mathbf{E}(Y_d^2)\mathbf{E}(Y_{n'})\mathbf{E}(Y_{m'}).$$

By the definition of Y_p, we have $\mathbf{E}(Y_{n'})=0$ if n' is divisible by a prime p with odd exponent, and similarly for $\mathbf{E}(Y_{m'})$. Hence we have $\mathbf{E}(Y_nY_m)=0$ unless both n' and m' are squares. Therefore

$$\mathbf{E}\Big(\Big|\sum_{n\leqslant u} Y_n n^{-\sigma}\Big|^2\Big) \leqslant \sum_{d\geqslant 1}\frac{\mathbf{E}(Y_d^2)}{d^{2\sigma}} \sum_{m,n\geqslant 1}\frac{1}{(mn)^{2\sigma}}\mathbf{E}(Y_mY_n) < +\infty$$

since $\sigma > 1/2$ and $\mathbf{E}(Y_n) \ll n^{\varepsilon}$ for any $\varepsilon > 0$. □

The key arithmetic properties of the family $S_2(q)^*$ of modular forms that are required in the proof of Theorem 2 are the following:

Proposition 1 (Local Spectral Equidistribution) *As $q \to +\infty$, the sequence $(\lambda_f(p))_p$ of Fourier coefficients of $f \in S_2(q)^*$ converges in law to the sequence $(Y_p)_p$.*

Proof This is a well-known consequence of the Petersson formula, see, e.g., [11, Proposition 8], [12, Appendix] or [3, Proposition 1.9]; here restricting to prime level q and weight 2 also simplifies matters since this ensures that the old space of $S_2(q)$ is zero. □

Proposition 2 (First Moment Estimate) *There exists an absolute constant $A \geqslant 1$ such that for any real number $\delta > 0$ with $\delta < 1/2$, and for any $s \in \mathbf{C}$ such that $\frac{1}{2}+\delta \leqslant \operatorname{Re}(s)$, we have*

$$\mathbf{E}_q(|L(f,s)|) \ll (1+|s|)^A,$$

where the implied constant depends only on δ.

Proof This follows easily, using the Cauchy–Schwarz inequality, from the second-moment estimate [10, Proposition 5] of Kowalski and Michel (with $\Delta = 0$); although this statement is not formally the same, it is in fact a more difficult average (it operates closer to the critical line). □

We now prove some additional lemmas.

Lemma 3 (Polynomial Growth) *For any real number $\sigma > \sigma_0$, we have*

$$\mathbf{E}\Big(\Big|\sum_{n\geqslant 1} Y_n n^{-s}\Big|\Big) \ll 1 + |s|$$

uniformly for all s such that $\mathrm{Re}(s) \geqslant \sigma > \sigma_0$.

Proof We write

$$L(s) = \sum_{n\geqslant 1} Y_n n^{-s}.$$

This is almost surely a function holomorphic on the half-plane S. The series

$$\sum_{n\geqslant 1} \frac{Y_n}{n^{\sigma_0}}$$

converges almost surely. Therefore the partial sums

$$S_u = \sum_{n\leqslant u} \frac{Y_n}{n^{\sigma_0}}$$

are bounded almost surely. By summation by parts, it follows from the convergence of the series $L(s)$ that for any s with real part $\mathrm{Re}(s) \geqslant \sigma > \sigma_0$, we have

$$L(s) = (s-\sigma_0)\int_1^{+\infty} \frac{S_u}{u^{s-\sigma_0+1}}du,$$

where the integral converges almost surely. Hence almost surely

$$|L(s)| \leqslant (1+|s|)\int_1^{+\infty} \frac{|S_u|}{u^{\sigma-\sigma_1+1}}du.$$

Fubini's Theorem and the Cauchy–Schwarz inequality then imply

$$\begin{aligned}\mathbf{E}(|L(s)|) &\leqslant (1+|s|)\int_1^{+\infty} \mathbf{E}(|S_u|)\frac{du}{u^{\sigma-\sigma_0+1}}\\ &\leqslant (1+|s|)\int_1^{+\infty} \mathbf{E}(|S_u|^2)^{1/2}\frac{du}{u^{\sigma-\sigma_0+1}} \ll 1+|s|\end{aligned}$$

by Lemma 2 (3). □

We now consider some elementary approximation statements of the L-functions and of the random Dirichlet series by smoothed partial sums. For this, we fix once and for all a smooth function $\varphi: [0, +\infty[\to [0, 1]$ with compact support such that $\varphi(0) = 1$, and we denote $\hat{\varphi}$ its Mellin transform.

We also fix $T \geqslant 1$ and a compact interval I in $]1/2, 1[$ such that the compact rectangle $R = I \times [-T, T] \subset \mathbf{C}$ is contained in S and contains D in its interior. We then finally define $\delta > 0$ so that

$$\min\{\mathrm{Re}(s) \mid s \in R\} = \frac{1}{2} + 2\delta.$$

Lemma 4 *For $N \geqslant 1$, define the $\mathcal{H}(D)$-valued random variable*

$$L_D^{(N)} = \sum_{n \geqslant 1} Y_n \varphi\Big(\frac{n}{N}\Big) n^{-s}.$$

We then have

$$\mathbf{E}(\|L_D - L_D^{(N)}\|_\infty) \ll N^{-\delta}$$

for $N \geqslant 1$, where the implied constant depends on D.

Proof We again write

$$L(s) = \sum_{n \geqslant 1} Y_n n^{-s}$$

when we wish to view the Dirichlet series as defined and holomorphic (almost surely) on S.

For any s in the rectangle R, we have almost surely the representation

$$L(s) - L^{(N)}(s) = -\frac{1}{2i\pi} \int_{(-\delta)} L(s + w) \hat{\varphi}(w) N^w dw \tag{4}$$

by standard contour integration.[2]

We also have almost surely for any v in D the Cauchy formula

$$L_D(v) - L_D^{(N)}(v) = \frac{1}{2i\pi} \int_{\partial R} (L(s) - L^{(N)}(s)) \frac{ds}{s - v},$$

[2]Here and below, it is important that the "almost surely" property holds for *all* s, which is the case because we work with random holomorphic functions, and not with particular evaluations of these random functions at specific points s.

where the boundary of R is oriented counterclockwise. The definition of the rectangle R ensures that $|s-v|^{-1} \gg 1$ for $v \in D$ and $s \in \partial R$, and therefore

$$\|L_D - L_D^{(N)}\|_\infty \ll \int_{\partial R} |L(s) - L^{(N)}(s)|\,|ds|.$$

Using (4) and writing $w = -\delta + iu$ with $u \in \mathbf{R}$, we obtain

$$\|L_D - L_D^{(N)}\|_\infty \ll N^{-\delta} \int_{\partial R}\int_{\mathbf{R}} |L(-\delta+iu+s)|\,|\hat{\varphi}(-\delta+iu)||ds|du.$$

Therefore, taking the expectation, and using Fubini's Theorem, we get

$$\begin{aligned}\mathbf{E}(\|L_D - L_D^{(N)}\|_\infty) &\ll N^{-\delta} \int_{\partial R}\int_{\mathbf{R}} \mathbf{E}\big(|L(-\delta+iu+s)|\big)\,|\hat{\varphi}(-\delta+iu)||ds|du \\ &\ll N^{-\delta} \sup_{s=\sigma+it\in R} \int_{\mathbf{R}} \mathbf{E}\big(|L(-\delta+iu+\sigma+it)|\big)\,|\hat{\varphi}(-\delta+iu)|du.\end{aligned}$$

We therefore need to bound

$$\int_{\mathbf{R}} \mathbf{E}\big(|L(-\delta+iu+\sigma+it)|\big)\,|\hat{\varphi}(-\delta+iu)|du.$$

for some fixed $\sigma + it$ in the compact rectangle R. The real part of the argument $-\delta + iu + \sigma + it$ is $\sigma - \delta \geqslant \frac{1}{2} + \delta$ by definition of δ, and hence

$$\mathbf{E}(|L(-\delta+iu+\sigma+it)|) \ll 1 + |-\delta+iu+\sigma+it| \ll 1 + |u|$$

uniformly for $\sigma + it \in R$ and $u \in \mathbf{R}$ by Lemma 3. Since $\hat{\varphi}$ decays faster than any polynomial at infinity, we conclude that

$$\int_{\mathbf{R}} \mathbf{E}\big(|L(-\delta+iu+\sigma+it)|\big)\,|\hat{\varphi}(-\delta+iu)|du \ll 1$$

uniformly for $s = \sigma + it \in R$, and the result follows. □

We proceed similarly for the L-functions.

Lemma 5 *For $N \geqslant 1$ and $f \in S_2(q)^*$, define*

$$L^{(N)}(f,s) = \sum_{n\geqslant 1} \lambda_f(n)\varphi\Big(\frac{n}{N}\Big)n^{-s},$$

and define $\mathsf{L}_q^{(N)}$ to be the $\mathcal{H}(D)$-valued random variable mapping f to $L^{(N)}(f,s)$ restricted to D. We then have

$$\mathbf{E}_q(\|\mathsf{L}_q - \mathsf{L}_q^{(N)}\|_\infty) \ll N^{-\delta}$$

for $N \geqslant 1$ and all q.

Proof For any $s \in R$, we have the representation

$$L(f,s) - L^{(N)}(f,s) = -\frac{1}{2i\pi}\int_{(-\delta)} L(f,s+w)\hat{\varphi}(w)N^w dw. \tag{5}$$

and for any v with $\mathrm{Re}(v) > 1/2$, Cauchy's theorem gives

$$L(f,v) - L^{(N)}(f,v) = \frac{1}{2i\pi}\int_{\partial R} (L(f,s) - L^{(N)}(f,s))\frac{ds}{s-v},$$

where the boundary of R is oriented counterclockwise. As in the previous argument, we deduce that

$$\|\mathsf{L}_q - \mathsf{L}_q^{(N)}\|_\infty \ll \int_{\partial R} |L(f,s) - L^{(N)}(f,s)||ds|$$

for $f \in S_2(q)^*$. Taking the expectation with respect to f and changing the order of summation and integration leads to

$$\begin{aligned}\mathbf{E}_q\Big(\|\mathsf{L}_q - \mathsf{L}_q^{(N)}\|_\infty\Big) &\ll \int_{\partial R} \mathbf{E}_q\big(|L(f,s) - L^{(N)}(f,s)|\big)|ds| \\ &\ll \sup_{s\in\partial R} \mathbf{E}_q\big(|L(f,s) - L^{(N)}(f,s)|\big). \end{aligned}\tag{6}$$

Applying (5) and using again Fubini's Theorem, we obtain

$$\mathbf{E}_q\big(|L(f,s) - L^{(N)}(f,s)|\big) \ll N^{-\delta}\int_{\mathbf{R}} |\hat{\varphi}(-\delta+iu)|\mathbf{E}_q\big(|L(f,-\delta+iu+\sigma+it)|\big)du$$

for $s \in \partial R$. Since $\sigma - \delta \geqslant \frac{1}{2} + \delta$, we get

$$\mathbf{E}_q\big(|L(f,-\delta+iu+\sigma+it)|\big) \ll (1+|u|)^A \tag{7}$$

by Proposition 2, where A is an absolute constant. Hence

$$\mathbf{E}_q\big(|L(f,s) - L^{(N)}(f,s)|\big) \ll N^{-\delta}\int_{\mathbf{R}} |\hat{\varphi}(-\delta+iu)|(1+|u|)^A du \ll N^{-\delta}. \tag{8}$$

We conclude from (6) that

$$\mathbf{E}_q\Big(\|\mathsf{L}_q - \mathsf{L}_q^{(N)}\|_\infty\Big) \ll N^{-\delta},$$

as claimed. □

Proof (Proof of Theorem 2) A simple consequence of the definition of convergence in law shows that it is enough to prove that for any bounded and Lipschitz function $f : \mathcal{H}(D) \longrightarrow \mathbf{C}$, we have

$$\mathbf{E}_q(f(\mathsf{L}_q)) \longrightarrow \mathbf{E}(f(L_D))$$

as $q \to +\infty$ (see [2, p. 16, (ii)$\Rightarrow$ (iii) and (1.1), p. 8]). To prove this, we use the Dirichlet series expansion of L_D given by Lemma 2 (2).

Let $N \geqslant 1$ be some integer to be chosen later. Let

$$\mathsf{L}_q^{(N)} = \sum_{n \geqslant 1} \lambda_f(n) n^{-s} \varphi\Big(\frac{n}{N}\Big)$$

(viewed as random variable defined on $S_2(q)^*$) and

$$L_N = \sum_{n \geqslant 1} Y_n n^{-s} \varphi\Big(\frac{n}{N}\Big)$$

be the smoothed partial sums of the Dirichlet series, as in Lemmas 5 and 4.

We then write

$$|\mathbf{E}_q(f(\mathsf{L}_q)) - \mathbf{E}(f(L))| \leqslant |\mathbf{E}_q(f(\mathsf{L}_q) - f(\mathsf{L}_q^{(N)}))| + |\mathbf{E}_q(f(\mathsf{L}_q^{(N)})) - \mathbf{E}(f(L^{(N)}))| + |\mathbf{E}(f(L^{(N)}) - f(L))|.$$

Since f is a Lipschitz function on $\mathcal{H}(D)$, there exists a constant $C \geqslant 0$ such that

$$|f(x) - f(y)| \leqslant C\|x - y\|_\infty$$

for all $x, y \in \mathcal{H}(D)$. Hence we have

$$|\mathbf{E}_q(f(\mathsf{L}_q)) - \mathbf{E}(f(L))| \leqslant C\mathbf{E}_q(\|\mathsf{L}_q - \mathsf{L}_q^{(N)}\|_\infty) + |\mathbf{E}_q(f(\mathsf{L}_q^{(N)})) - \mathbf{E}(f(L^{(N)}))| + C\mathbf{E}(\|L^{(N)} - L\|_\infty).$$

Fix $\varepsilon > 0$. Lemmas 5 and 4 together show that there exists some $N \geqslant 1$ such that

$$\mathbf{E}_q(\|\mathsf{L}_q - \mathsf{L}_q^{(N)}\|_\infty) < \varepsilon$$

for all $q \geqslant 2$ and

$$\mathbf{E}(\|L^{(N)} - L\|_\infty) < \varepsilon.$$

We *fix* such a value of N. By Proposition 1 (and composition with a continuous function), the random variables $\mathsf{L}_q^{(N)}$ (which are Dirichlet polynomials) converge in law to $L^{(N)}$ as $q \to +\infty$. We deduce that we have

$$|\mathbf{E}_q(f(\mathsf{L}_q)) - \mathbf{E}(f(L))| < 4\varepsilon$$

for all q large enough. This finishes the proof. □

4 Proof of Theorem 3

For the computation of the support of the random Dirichlet series $L(s)$, we apply a trick to exploit the analogous result known for the case of the Riemann zeta function. We denote $\widehat{SU}_2$ the product of copies of the unit circle indexed by primes, so an element (x_p) of $\widehat{SU}_2$ is a family of matrices in $\mathrm{SU}_2(\mathbf{C})$ indexed by p.

The assumptions on D in Theorem 3[3] imply that there exists τ such that $1/2 < \tau < 1$ and $r > 0$ such that

$$D = \{s \in \mathbf{C} \mid |s - \tau| \leqslant r\} \subset \{s \in \mathbf{C} \mid 1/2 < \mathrm{Re}(s) < 1\}.$$

Lemma 6 *Let N be an arbitrary positive real number. The set of all series*

$$\sum_{p>N} \frac{\mathrm{Tr}(x_p)}{p^s}, \qquad (x_p) \in \widehat{SU}_2$$

which converge in $\mathcal{H}(D)$ is dense in the subspace $\mathcal{H}_{\mathbf{R}}(D)$.

In the proof and the next, we allow ourselves the luxury of writing sometimes $\|\varphi(s)\|_\infty$ instead of $\|\varphi\|_\infty$.

Proof Bagchi [1, Lemma 5.2.10] proves (using results of complex analysis due to Bernstein, Polyá, and others) that the set of series

$$\sum_{p>N} \frac{e^{i\theta_p}}{p^s}, \qquad \theta_p \in \mathbf{R}$$

that converge in $\mathcal{H}(D)$ is dense in $\mathcal{H}(D)$ (precisely, he proves this for $N = 1$, but the same proof applies to any value of N). If $\varphi \in \mathcal{H}_{\mathbf{R}}(D)$ and $\varepsilon > 0$, we can therefore find real numbers (θ_p) such that

$$\left\| \frac{\varphi(s)}{2} - \sum_{p>N} \frac{e^{i\theta_p}}{p^s} \right\|_\infty < \frac{\varepsilon}{2}.$$

[3]These assumptions could be easily weakened, as has been done for Voronin's Theorem.

It follows then that

$$\left\| \overline{\frac{\varphi(\bar{s})}{2}} - \sum_{p>N} \frac{e^{-i\theta_p}}{p^s}, \right\|_\infty < \frac{\varepsilon}{2},$$

hence (since $\varphi \in \mathcal{H}_{\mathbf{R}}(D)$) that

$$\left\| \varphi(s) - \sum_{p>N} \frac{e^{i\theta_p} + e^{-i\theta_p}}{p^s} \right\|_\infty < \varepsilon,$$

which gives the result since

$$e^{i\theta_p} + e^{-i\theta_p} = \mathrm{Tr}\begin{pmatrix} e^{i\theta_p} & 0 \\ 0 & e^{-i\theta_p} \end{pmatrix}$$

is the trace of a matrix in $\mathrm{SU}_2(\mathbf{C})$. □

We will use this to prove:

Proposition 3 *The support of the law of*

$$\log L_D(s) = -\sum_p \log\det(1 - X_p p^{-s})$$

in $\mathcal{H}(D)$ *is* $\mathcal{H}_{\mathbf{R}}(D)$.

Proof Since $X_p \in \mathrm{SU}_2(\mathbf{C})$, the function $\log L(s)$ is almost surely in the space $\mathcal{H}_{\mathbf{R}}(D)$. Since the summands are independent, a well-known result concerning the support of random series (see, e.g., [9, Proposition B.8.7]) shows that it suffices to prove that the set of convergent series

$$-\sum_p \log\det(1 - x_p p^{-s}), \qquad (x_p) \in \widehat{SU}_2,$$

is dense in $\mathcal{H}_{\mathbf{R}}(D)$. Denote $L(s;(x_p))$ this series, when it converges in $\mathcal{H}(D)$.

We can write

$$-\sum_p \log\det(1 - x_p p^{-s}) = \sum_p \frac{\mathrm{Tr}(x_p)}{p^s} + g(s;(x_p))$$

where $s \mapsto g(s;(x_p))$ is holomorphic in the region $\mathrm{Re}(s) > 1/2$. Indeed

$$g(s;(x_p)) = \sum_p \log\Big(1 + \sum_{k\geqslant 0} \mathrm{Tr}(x_p)^k p^{-(k+2)s}\Big).$$

Fix $\varphi \in \mathcal{H}_{\mathbf{R}}(D)$ and let $\varepsilon > 0$ be fixed. There exists $N \geqslant 1$ such that

$$\left\| s \mapsto \sum_{p>N} \log\Big(1 + \sum_{k\geqslant 0} \mathrm{Tr}(x_p)^k p^{-(k+2)s}\Big) \right\|_\infty < \varepsilon \tag{9}$$

for *any* $(x_p) \in \widehat{SU}_2$. Now take $x_p = 1 \in \mathrm{SU}_2(\mathbf{C})$ for $p \leqslant N$ and define

$$\varphi_1 = \varphi + \sum_{p\leqslant N} \frac{\mathrm{Tr}(x_p)}{p^s} = \varphi + 2\sum_{p\leqslant N} \frac{1}{p^s},$$

which belongs to $\mathcal{H}_{\mathbf{R}}(D)$. By Lemma 6, there exist x_p for $p > N$ in $\mathrm{SU}_2(\mathbf{C})$ such that

$$\left\| \sum_{p>N} \frac{\mathrm{Tr}(x_p)}{p^s} - \varphi_1(s) \right\|_\infty < \varepsilon.$$

The left-hand side is the norm of

$$\log L(s;(x_p)) - g(s;(x_p)) - \sum_{p\leqslant N} \frac{\mathrm{Tr}(x_p)}{p^s} - \varphi_1(s) = \log L(s;(x_p)) - \varphi(s) - g(x;(x_p)),$$

and by (9), we obtain

$$\| \log L(s;(x_p)) - \varphi(s) \|_\infty < 2\varepsilon.$$

This implies the lemma. □

Using composition with the exponential function and a lemma of Hurwitz (see, e.g., [17, 3.45]) on zeros of limits of holomorphic functions, we see that the support of the limiting Dirichlet series L_D in $\mathcal{H}(D)$ is the union of the zero function and the set of functions $\varphi \in \mathcal{H}_{\mathbf{R}}(D)$ such that $\varphi(\sigma) > 0$ for $\sigma \in D \cap \mathbf{R}$. In particular, this proves Theorem 3.

5 Generalizations

It is clear from the proof that Bagchi's Theorem should hold in considerable generality for any family of L-functions. Indeed, the crucial ingredients are the local spectral equidistribution (Proposition 1) and the first moment estimate (Proposition 2).

The first result is a qualitative statement that is understood to be at the core of any definition of "family" of L-functions (this is explained in [8], but also appears, with a different terminology, for the families of Conrey et al. [5] and Sarnak et al. [15]); it is now know in many circumstances (indeed, often in quantitative form).

The moment estimate is typically derived from a second-moment bound, and is also definitely expected to hold for a reasonable family of L-functions, but it has only been proved in much more restricted circumstances than local spectral equidistribution. However, it is very often the case that one can at least prove (using local spectral equidistribution) a weaker statement: for some σ_1 such that $1/2 < \sigma_1 < 1$, the second moment of the L-functions satisfies the analogue of Proposition 2; an analogue of Bagchi's Theorem then follows at least for compact discs in the region $\sigma_1 < \mathrm{Re}(s) < 1$.

As far as universality (i.e., Theorem 3) is concerned, one may expect that (using tricks similar to the proof of Theorem 3) only two different cases really occur, depending on whether the coefficients of the L-functions are real (as in our case) or complex (as in the case of vertical translates of a fixed L-function).

Acknowledgements This work was partially supported by a DFG-SNF lead agency program grant (grant 200021L_153647).

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The Liouville Function and the Riemann Hypothesis

Michael J. Mossinghoff and Timothy S. Trudgian

Abstract For nearly a century, mathematicians have explored connections between the Liouville function and the Riemann hypothesis. We describe a number of connections regarding oscillations in sums involving the Liouville function, including $L_0(x) = \sum_{n \le x} \lambda(n)$, first studied by Pólya, and $L_1(x) = \sum_{n \le x} \lambda(n)/n$, explored by Turán. We establish new lower bounds on the size of the oscillations in such sums. In particular, we prove that $L_0(x) > \sqrt{x}$ infinitely often, show that $L_1(x) < -1/\sqrt{x}$ infinitely often, and investigate sign changes in a family of functions that interpolate these functions, namely $L_\alpha(x) = \sum_{n \le x} \lambda(n)/n^\alpha$, where $\alpha \in [0, 1]$.

1 Introduction

The Liouville function $\lambda(n)$ is the completely multiplicative arithmetic function whose value is -1 at each prime, so $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the number of prime factors of n, counting multiplicity. For nearly 100 years mathematicians have explored connections between this function and the Riemann hypothesis. Ultimately, these are connected by the Dirichlet series for the Liouville function: for $\Re(s) > 1$, using the Euler product representation of the zeta function we obtain

$$\frac{\zeta(2s)}{\zeta(s)} = \frac{\prod_p (1 - p^{-2s})^{-1}}{\prod_p (1 - p^{-s})^{-1}} = \prod_p (1 + p^{-s})^{-1} = \prod_p \sum_{k \ge 0} \frac{(-1)^k}{p^{ks}} = \sum_{n \ge 1} \frac{\lambda(n)}{n^s}.$$

Here and throughout this article, s denotes a complex variable with real part σ and imaginary part t.

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H. Montgomery et al. (eds.), *Exploring the Riemann Zeta Function*,
DOI 10.1007/978-3-319-59969-4_9

For a nonnegative real number α, let $L_\alpha(x)$ denote the following weighted sum:

$$L_\alpha(x) = \sum_{n \le x} \frac{\lambda(n)}{n^\alpha}. \tag{1}$$

The two special cases $\alpha = 0$ and $\alpha = 1$ have a long history in analytic number theory, in topics surrounding the Riemann hypothesis.

Pólya investigated the sum $L_0(x)$ [which often appears in the literature as $L(x)$] in a 1919 publication [27]. His main result was a proof that if $p > 7$ is a prime with $p \equiv 3 \bmod 4$, such that the quadratic imaginary number field $\mathbb{Q}(\sqrt{-p})$ has class number 1, then $L_0(x)$ must vanish at $x = (p-3)/4$. He then reported computing $L_0(x)$ up to approximately 1500, and noted that it was never positive over this range, after the trivial value $L_0(1) = 1$. Pólya wrote:

> I mention this observation, in order to prompt perhaps further numerical research. The proof of (1) [referring to $L_0(n) \le 0$], even only for sufficiently large n, would produce a proof of the Riemann hypothesis, after known properties of the ζ-function and a function-theoretic result of Mr. Landau.

The statement that $L_0(n) \le 0$ for sufficiently large n is often described in the literature as *Pólya's conjecture*, but in fact Pólya only noted this implication, and did not conjecture its validity, at least in print. We therefore employ the term *Pólya's problem* in this article for the question of sign changes, and more broadly of oscillations, in the function $L_0(x)$.

Turán studied the weighted sum $L_1(x)$ [which often appears in the literature as $T(x)$] in 1948 [31], in connection with the zeros of the truncated series for the zeta function, $U_x(s) = \sum_{n \le x} 1/n^s$. Turán proved that if there exists a positive constant K such that $L_1(x) > -K/\sqrt{x}$ for sufficiently large x, then $U_x(s)$ does not vanish in the half plane $\sigma \ge 1 + K/\sqrt{x}$, which would imply the Riemann hypothesis. (Somewhat weaker conditions suffice as well; see [31, 32].) Turán reported that a coterie of Danish colleagues computed $L_1(x)$ for $x \le 1000$, and found that this function remained positive throughout this range. Perhaps because of this, in later literature the statement that $L_1(x)$ is never negative is sometimes stated as *Turán's conjecture*. However, in a later paper [32] Turán took issue with the term:

> Strangely, this [referring to the positivity of $L_1(n)$] is described in subsequent work as "Turán's conjecture," even claimed for $n \ge 1$, but this occurs nowhere in [31], for $n \ge 1$ not even implicitly claimed.

We use the term *Turán's problem* in this article to refer to the question of sign changes and oscillations in $L_1(x)$.

Turán's original motivation for studying this problem was settled by Montgomery in 1983 [22], who proved that $U_x(n)$ does have zeros some distance to the right of the $\sigma = 1$ line in the complex plane. He proved that for every positive constant $c < \frac{4}{\pi} - 1$ and every integer $x > x_0(c)$ the function $U_x(s)$ has a zero in the half plane $\sigma > 1 + c \log\log x / \log x$.

A good deal is known about both Pólya's and Turán's problem. In 1958, Haselgrove [11] proved that both $L_0(x)$ and $L_1(x)$ change sign infinitely often.

Lehman [19] first determined an explicit location where $L_0(x)$ is positive, proving in 1960 that $L_0(906\,180\,359) = 1$. The location of the first sign crossing of $L_0(x)$ for $x \geq 2$ was not found until 1980, when Tanaka [30] established that this occurs at $x = 906\,150\,257$. In Turán's problem, the first sign crossing was found more recently by Borwein, Ferguson, and the first author [5]. It occurs at $x = 72\,185\,376\,951\,205$.

It is well known (and was certainly known to Pólya and Turán) that the Riemann hypothesis would follow if one could bound the oscillations in $L_0(x)$ or $L_1(x)$. The Riemann hypothesis, as well as the simplicity of the zeros of $\zeta(s)$, would follow if there exists a positive constant C such that any one of the following inequalities holds for sufficiently large x:

$$\frac{L_0(x)}{\sqrt{x}} < C, \ \frac{L_0(x)}{\sqrt{x}} > -C, \ L_1(x)\sqrt{x} < C, \ L_1(x)\sqrt{x} > -C. \tag{2}$$

This follows from Landau's theorem regarding singularities of the Laplace transform of a nonnegative function (see, for instance, [23, Lemma 15.1]).

It is easy to see that bounding $L_0(x)/\sqrt{x}$ on both sides would yield the Riemann hypothesis. Starting with the Dirichlet series for the Liouville function in the complex variable $s = \sigma + it$, one may compute

$$\begin{aligned} \frac{\zeta(2s)}{\zeta(s)} &= \sum_{n\geq 1} \frac{\lambda(n)}{n^s} = \sum_{n\geq 1} \frac{L_0(n) - L_0(n-1)}{n^s} = \sum_{n\geq 1} L_0(n) \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \\ &= \sum_{n\geq 1} L_0(n) \int_n^{n+1} \frac{s}{x^{s+1}}\, dx = s \int_1^{\infty} \frac{L_0(x)}{x^{s+1}}\, dx, \end{aligned} \tag{3}$$

so that clearly $L_0(x) = O(\sqrt{x})$ implies that the right side of (3) is analytic on $\sigma > 1/2$, and therefore so is the left side, which would establish the Riemann hypothesis. One can deduce that the zeros of the zeta function on the critical line are all simple in this case as well.

A result of Ingham [14] in 1942 cast some doubt on the possibility of proving the Riemann hypothesis and the simplicity of the zeros of $\zeta(s)$ by bounding $L_0(x)/\sqrt{x}$ or similar functions. Ingham proved that such a result would imply considerably more: it would follow that infinitely many integer relations exist among the ordinates of the zeros of $\zeta(s)$ on the critical line in the upper half plane. Since there seems to be no other particular reason to suspect the existence of such relations, and no evidence of such relations has been detected, it seems plausible that these (suitably scaled) sums involving the Liouville function may be unbounded. In this article, we focus on obtaining new and larger bounds on these oscillations.

Some prior bounds on oscillations for $L_0(x)$ are known. Anderson and Stark [2] proved that single values of $L_0(x)$ could be employed to quantify the oscillations of $L_0(x)/\sqrt{x}$. They showed that

$$\limsup_{x\to\infty} \frac{L_0(x)}{\sqrt{x}} \geq \frac{L_0(x_0) - I(x_0)}{\sqrt{x_0}}, \tag{4}$$

where $I(x)$ satisfies $|I(x) - 1| \leq \frac{2\sqrt{2}\pi\zeta(3/2)}{3\zeta(3)\sqrt{x}} \approx 6.437/\sqrt{x}$. Lehman [19] reported that $L_0(906\,400\,000) = 708$, whence by (4), $L_0(x) > 0.0234833\sqrt{x}$ infinitely often. Tanaka [30] found that $L_0(x)$ reaches a maximum value of 829 for $x \leq 10^9$, and while he did not report that this value occurred at $x = 906\,316\,571$, with this datum (4) shows that $L_0(x) > 0.0275036\sqrt{x}$ infinitely often. The more recent work of Borwein, Ferguson, and the first author [5] determined that $L_0(351\,753\,358\,289\,465) = 1{,}160{,}327$, which establishes that

$$L_0(x) > 0.0618672\sqrt{x}$$

infinitely often. Further, as noted by Humphries [13], the method of Anderson and Stark also shows that $\liminf_{x\to\infty} L_0(x)/\sqrt{x} \leq (L_0(x_0) - I(x_0))/\sqrt{x_0}$ for any particular x_0. The data point $L_0(72\,204\,113\,780\,255) = -11\,805\,117$ from [5] establishes that

$$L_0(x) < -1.389278\sqrt{x}$$

infinitely often.

In this article, we prove even larger oscillations in $L_0(x)$. Due to some symmetry in these problems, our results apply equally to $L_1(x)$, after suitable scaling. We prove the following result in Sect. 6.

Theorem 1 *Each of the following inequalities is satisfied for infinitely many integers x:*

$$L_0(x) > 1.0028\sqrt{x}, \quad L_0(x) < -2.3723\sqrt{x},$$
$$L_1(x) < -1.0028/\sqrt{x}, \quad L_1(x) > 2.3723/\sqrt{x}.$$

The more general family $L_\alpha(x)$ with $\alpha \in [0, 1]$ was studied by the authors [24], by Humphries [13], and by Akbary et al. [1]. It will be seen in Sect. 2 that if $L_\alpha(x)$ eventually has no sign changes for some $\alpha \in [0, 1/2)$, or if $L_\alpha(x) - \zeta(2\alpha)/\zeta(\alpha)$ eventually has no sign changes for some $\alpha \in (1/2, 1]$, then the Riemann hypothesis follows, as well as the simplicity of the zeros, and the existence of infinitely many integer relations among the ordinates of the zeros of the zeta function in the upper half plane. Because of this, in [24] the authors raised the question of whether these functions do change sign infinitely often. Our methods in this article allow us to resolve this question for almost 60% of the interval $[0, 1]$. We establish the following theorem in Sect. 6.

Theorem 2 *For each $\alpha \in [0, 0.29714\ldots)$, the function $L_\alpha(x)$ exhibits infinitely many sign changes, and the same conclusion holds for each $\alpha \in (0.70285\ldots, 1]$ for the function $L_\alpha(x) - \zeta(2\alpha)/\zeta(\alpha)$.*

Some additional explicit results regarding the oscillations of $L_\alpha(x)$ for particular values of α appear in Sect. 6 (see Theorem 8).

This article is organized in the following way. Section 2 describes a normalization of the functions $L_\alpha(x)$ that allows us to treat their oscillations in a uniform way. Section 3 is an interlude on sums involving the Möbius function. Section 4 reports on a method involving the so-called weak independence of a set of real numbers, which quantifies oscillations in sums of arithmetic functions. Section 5 then describes a more recent strategy developed by Best and the second author [4] to obtain improved bounds on oscillations for sums involving the Möbius function. Section 6 describes an adaptation of this method to sums involving the Liouville function, and establishes Theorems 1 and 2. Finally, Sect. 7 briefly discusses the potential for future improvements.

2 Generalizing the Problems of Pólya and Turán

We begin by defining some functions that will allow a more uniform treatment of oscillation questions for the functions $L_\alpha(x)$ with $\alpha \in [0,1]$. In [24], it is shown that the Riemann hypothesis is true if and only if $L_\alpha(x)$ converges to $\zeta(2\alpha)/\zeta(\alpha)$ for each $\alpha \in (1/2,1]$. In fact, convergence at a particular α in this range is tantamount to showing that the zeta function has no zeros in the half plane $\sigma \geq \alpha$, so in particular the fact that $L_1(x)$ converges to 0 is equivalent to the Prime Number Theorem. Moreover, it is shown in [24] that the Riemann hypothesis produces an error bound of $O(x^{\frac{1}{2}-\alpha+\epsilon})$ for the quantity $L_\alpha(x)-\zeta(2\alpha)/\zeta(\alpha)$. A similar statement is shown to be equivalent to the Riemann hypothesis for the case of small α: for each $\alpha \in [0,1/2)$, this conjecture is equivalent to the statement that $L_\alpha(x) = O(x^{\frac{1}{2}-\alpha+\epsilon})$. We describe the special case $\alpha = 1/2$ shortly, but following [24] we define

$$\mathscr{L}_\alpha(x) = \begin{cases} L_\alpha(x), & \text{if } 0 \leq \alpha < 1/2 \text{ or } \alpha = 1, \\ L_\alpha(x) - \dfrac{\log x}{2\zeta(1/2)}, & \text{if } \alpha = 1/2, \\ L_\alpha(x) - \dfrac{\zeta(2\alpha)}{\zeta(\alpha)}, & \text{if } 1/2 < \alpha < 1, \end{cases} \tag{5}$$

and we let

$$A_\alpha(u) = \mathscr{L}_\alpha(e^u)e^{(\alpha-\frac{1}{2})u}. \tag{6}$$

Here, we have simply scaled $\mathscr{L}_\alpha(x)$ by the main term of the error bound (on the Riemann hypothesis), and substituted e^u for x for convenience. We may now state the result from [24] that generalizes (2).

Theorem 3 *Suppose $\alpha \in [0,1]$ is a fixed real number. If there exists a constant C such that $A_\alpha(u) - C$ has constant sign for all sufficiently large u, then the Riemann hypothesis follows, and all of the zeros of the zeta function are simple.*

It is therefore natural to study oscillations in the family of functions $A_\alpha(u)$ with $0 \le \alpha \le 1$. We exhibit some of these functions in Fig. 1. Parts (a) and (b) of this figure show $L_{1/4}(e^u)$ and $A_{1/4}(u)$, respectively, over the range $22 \le u \le 12 \log 10 \approx 27.6$. The horizontal line in part (b) shows the center of the oscillations of $A_{1/4}(u)$. We shall see in (9) below that this line occurs at $y = 2/\zeta(1/2) = -1.3695\ldots$. The plot for the corresponding function in Pólya's problem, $A_0(u)$, exhibits similar oscillations, but they are centered at $y = 1/\zeta(1/2) = -0.6847\ldots$. Thus, a sign change in $A_{1/4}(u)$ requires an upward oscillation that is twice as high as what is required for a sign change in Pólya's problem.

Figure 1 parts (c) and (d) illustrate the transitional phase, $\alpha = 1/2$: in (c), the oscillations center on the line $u/2\zeta(1/2) - \gamma/\zeta(1/2) + \zeta'(1/2)/2\zeta(1/2)^2$, as we shall see in (9). Last, parts (e) and (f) exhibit $L_{3/4}(e^u)$ and $A_{3/4}(u)$, respectively, and the positive bias with respect to the limit point is evident here. In part (e), the horizontal line shows the expected limiting value, $\zeta(3/2)/\zeta(3/4) = -0.7591\ldots$, and in part (f), the horizontal line shows the center of the oscillations, which in this case occurs at $-2/\zeta(1/2) = 1.3695\ldots$, again using (9) below. In Turán's problem, the oscillations in $A_1(u)$ would center on half this value.

In order to analyze the oscillations in the functions $A_\alpha(u)$, we require some additional definitions. Let $F_\alpha(s)$ denote the Laplace transform of $A_\alpha(u)$,

$$F_\alpha(s) = \int_0^\infty A_\alpha(u) e^{-su}\, du. \tag{7}$$

It is shown in [24] that $F_\alpha(s)$ converges unconditionally in the half plane $\sigma > 1/2$ and converges in $\sigma > 0$ on the Riemann hypothesis. The following formula is derived there:

$$F_\alpha(s) = \begin{cases} f_\alpha(s), & \text{if } 0 \le \alpha < 1/2 \text{ or } \alpha = 1, \\ f_\alpha(s) - \dfrac{1}{2\zeta(1/2)s^2}, & \text{if } \alpha = 1/2, \\ f_\alpha(s) - \dfrac{\zeta(2\alpha)}{\zeta(\alpha)(s-\alpha+1/2)}, & \text{if } 1/2 < \alpha < 1, \end{cases} \tag{8}$$

where

$$f_\alpha(s) = \frac{\zeta(1+2s)}{(s-\alpha+1/2)\zeta(s+1/2)}.$$

Notice that the term subtracted from $f_\alpha(s)$ to create $F_\alpha(s)$ eliminates the simple pole on the real axis at $\sigma = \zeta(2\alpha)/\zeta(\alpha)$ when $\alpha \in (1/2, 1)$, and removes the second-order pole at $\sigma = 0$ when $\alpha = 1/2$. This is the source of the additional term in (5) for the case $\alpha = 1/2$: we must ensure $F_\alpha(s)$ has only simple zeros in $\sigma \ge 0$ in order to apply Ingham's method.

At this point, and in the remainder of this article, we may assume the Riemann hypothesis and the simplicity of the zeros of the zeta function. This is permissible because it follows from Landau's theorem that if either the Riemann hypothesis is

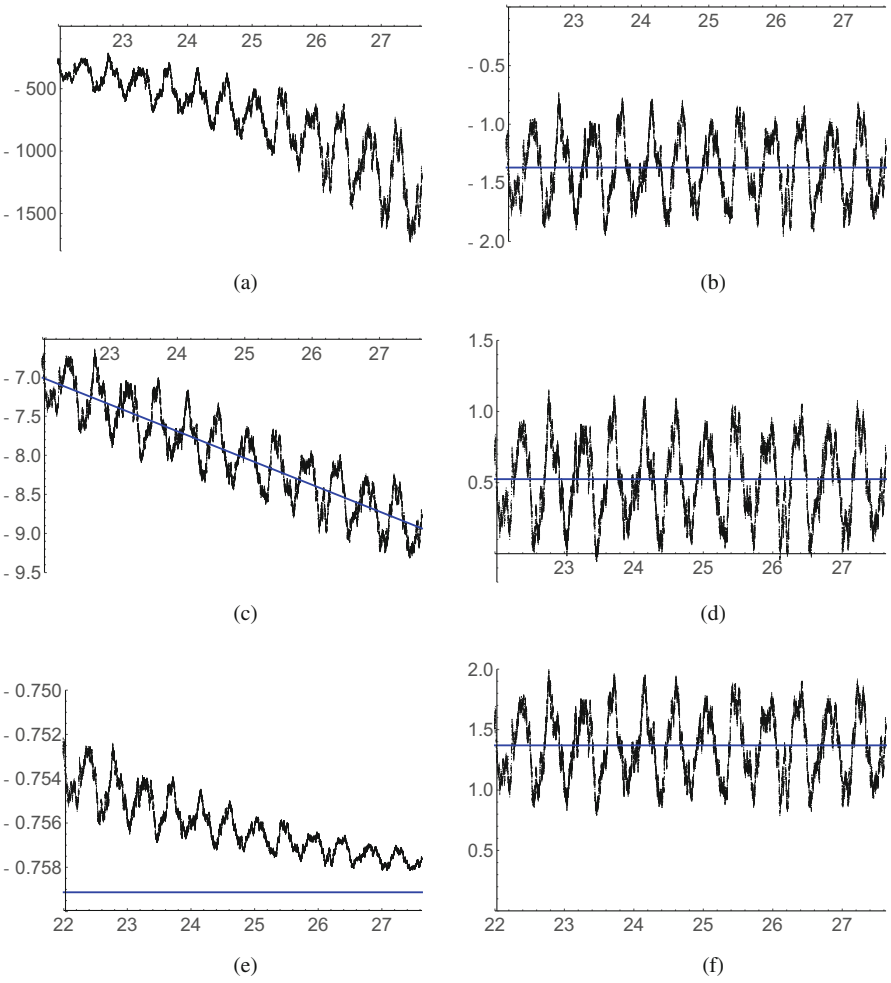

Fig. 1 $L_\alpha(e^u)$ and $A_\alpha(u)$ for $\alpha = 1/4,\ 1/2$, and $3/4$. (**a**) $L_{1/4}(e^u)$. (**b**) $A_{1/4}(u)$. (**c**) $L_{1/2}(e^u)$. (**d**) $A_{1/2}(u)$. (**e**) $L_{3/4}(e^u)$. (**f**) $A_{3/4}(u)$

false or $\zeta(s)$ has a multiple zero on the critical line, then $\limsup_{u\to\infty} A_\alpha(u) = \infty$ and $\liminf_{u\to\infty} A_\alpha(u) = -\infty$ (see [9, Sect. 8], [14, p. 318], and [13, Theorems 2.6, 2.7]). With this, we then have that $F_\alpha(s)$ is analytic in $\sigma \geq 0$, except for a number of simple poles on the imaginary axis. We need to record the residues of these poles. At the origin, we find

$$\mathrm{Res}(F_\alpha, 0) = \begin{cases} \dfrac{1}{(1-2\alpha)\zeta(1/2)}, & \text{if } \alpha \in [0, 1/2) \cup (1/2, 1], \\ \dfrac{\gamma}{\zeta(1/2)} - \dfrac{\zeta'(1/2)}{2\zeta(1/2)^2}, & \text{if } \alpha = 1/2, \end{cases} \tag{9}$$

where γ denotes Euler's constant, and if $\rho_n = 1/2 + i\gamma_n$ is a zero of the zeta function on the critical line, then

$$\operatorname{Res}(F_\alpha, i\gamma_n) = \frac{\zeta(2\rho_n)}{(\rho_n - \alpha)\zeta'(\rho_n)}, \quad 0 \le \alpha \le 1. \tag{10}$$

Ingham's method now applies. Select a positive real number T, and let $\gamma_1, \ldots, \gamma_N$ denote the ordinates of the zeros of the zeta function in the upper half plane with height at most T on the critical line. Let

$$F_\alpha^*(s) = \frac{\operatorname{Res}(F_\alpha, 0)}{s} + \sum_{|n|=1}^{N} \frac{\operatorname{Res}(F_\alpha, i\gamma_n)}{s - i\gamma_n}$$

so that $F_\alpha(s) - F_\alpha^*(s)$ is analytic in the region $\sigma \ge 0, t \in [-T, T]$. Let $k_T(t)$ be an even function supported on $[-T, T]$ with $k_T(0) = 1$, so that $k_T(t)$ is the Fourier transform of a nonnegative function $K_T(y)$. Let $B_\alpha^*(u)$ denote the inverse Laplace transform of $F_\alpha^*(s)$, except that each term is modified by a value of the weight function $k_T(t)$:

$$B_\alpha^*(u) = \operatorname{Res}(F_\alpha, 0) + 2\Re \sum_{0 < \gamma_n < T} k_T(\gamma_n) \operatorname{Res}(F_\alpha, i\gamma_n) e^{i\gamma_n u}. \tag{11}$$

Ingham's method then establishes the following inequalities. A proof may be found in [24].

Theorem 4 *Let $\alpha \in [0, 1]$ and $T > 1$ be fixed real numbers. Then, using the notation above,*

$$\liminf_{u\to\infty} A_\alpha(u) \le \liminf_{u\to\infty} B_\alpha^*(u) \le B_\alpha^*(v) \le \limsup_{u\to\infty} B_\alpha^*(u) \le \limsup_{u\to\infty} A_\alpha(u),$$

for any positive real number v.

Thus, to establish that the functions $A_\alpha(u)$ oscillate to some large degree, it is enough to show that the function $B_\alpha^*(v)$ achieves particularly large values, and achieves particularly small values. This is a much more attractive setting because $B_\alpha^*(v)$ is a finite sum. Haselgrove [11] used this approach to establish that infinitely many sign changes occur in both Pólya's and Turán's problem. Using $T = 1000$, he computed that $B_0^*(831.846)$ is positive, and that $B_1^*(853.853)$ is negative. Lehman [19] and Borwein et al. [5] also employed these functions to determine locations where $L_0(x)$ or $L_1(x)$ appeared likely to change sign.

We remark that the term corresponding to the first zero of the zeta function on the critical line, $\gamma_1 = 14.1347\ldots$, drives the overall periodic structure of $A_\alpha(u)$. Figure 2 illustrates this with a plot of $A_{1/4}(u)$ as in Fig. 1b overlaid with just the first two terms of (11) (without the weight function), $2/\zeta(1/2) + 2\Re(\operatorname{Res}(F_{1/4}, i\gamma_1)e^{i\gamma_1 u})$.

We may now describe Ingham's result in a more general setting. This argument has two principal ingredients. First, using methods dating to Titchmarsh, one shows that the sum of the moduli of the residues of $F_\alpha(s)$ at its poles on the positive

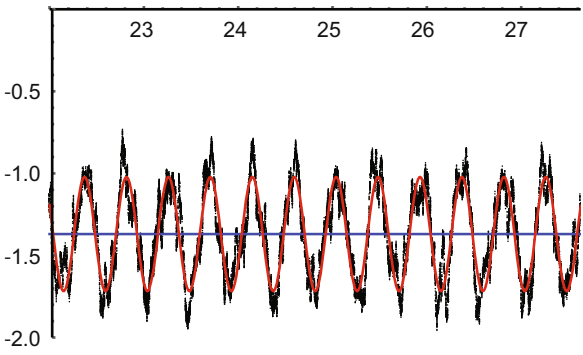

Fig. 2 $A_{1/4}(u)$ and its approximation using only the constant term and the single periodic term corresponding to the first zero of the zeta function on the critical line

imaginary axis diverges: $\sum_{n\geq 1} |\text{Res}(F_\alpha, i\gamma_n)| = \infty$. Second, if the ordinates of the zeros of the zeta function in the upper half plane are linearly independent, or if there are only finitely many linear dependencies, then by Kronecker's theorem on simultaneous approximation there exist infinitely many choices for v so that the complex argument of $\text{Res}(F_\alpha, i\gamma_n)e^{i\gamma_n v}$ is less than an arbitrarily selected $\epsilon > 0$ in absolute value, for each $\gamma_n \in (0, T)$. Similarly, there would also exist infinitely many choices of v so that the argument of each of these terms is within ϵ of π. Thus, on independence of the γ_n, by using (11) and the divergence of the sum of the residues we see that $B^*_\alpha(v)$ must achieve values that are as large as we like, in either the positive or the negative direction. It follows that $A_\alpha(u)$ would have unbounded oscillations. We state this below; a proof may be found in [24].

Theorem 5 *Let $\alpha \in [0, 1]$. If the imaginary parts $\gamma_1 < \gamma_2 < \cdots$ of the distinct zeros of $\zeta(s)$ lying above the real axis satisfy at most a finite number of linear relations of the type $\sum_{n=1}^{N} c_n\gamma_n = 0$, where the c_n are integers, not all zero, then $\liminf_{u\to\infty} A_\alpha(u) = -\infty$ and $\limsup_{u\to\infty} A_\alpha(u) = \infty$.*

3 The Möbius Function and the Riemann Hypothesis

The Möbius function, defined by $\mu(n) = \lambda(n)$ if n is squarefree and $\mu(n) = 0$ otherwise, also has a long history in analytic number theory in connection with the Riemann hypothesis. We remark here on a few analogues of the statements described earlier for the Liouville function.

For a real number $\alpha \in [0, 1]$, we may define

$$M_\alpha(x) = \sum_{n\leq x} \frac{\mu(n)}{n^\alpha}. \tag{12}$$

The cases $\alpha = 0$ and $\alpha = 1$ have attracted much interest for at least 130 years. Stieltjes [29] noted the importance of the function $M_1(x)$ in his letter to Hermite

in 1885. Von Mangoldt [33] proved that $\lim_{x\to\infty} M_1(x) = 0$ in 1897, and Landau established another proof of this fact in his 1899 dissertation [17]. A few years later, Landau [18] noted that the Prime Number Theorem was equivalent to the statement that $M_1(x) \to 0$ as $x \to \infty$, in that each of these statements can be derived from the other by using only elementary means.

The function $M_0(x)$ (known in the literature as the *Mertens function*, and often denoted by $M(x)$) has a particularly interesting history in analytic and computational number theory. It has long been known that the Riemann hypothesis and the simplicity of the zeros of $\zeta(s)$ would follow from a bound on the oscillations of $M_0(x)$, for example, the argument shown in (3) works just as well for the Dirichlet series for the Möbius function, $\sum_{n\geq 1} \mu(n)/n^s = 1/\zeta(s)$. In fact, Landau's theorem shows that these statements follow from any of the bounds analogous to those shown in (2), for example, it would be enough to show the existence of a constant C such that $M_0(x)/\sqrt{x} < C$ for all sufficiently large x. Motivated by some of these connections, Mertens computed $M_0(x)$ for $x \leq 10000$ and conjectured that the inequality $M_0(x) < \sqrt{x}$ for all $x > 1$ was "very probable." This *Mertens conjecture* remained open until 1985, when it was disproved by Odlyzko and te Riele [25].

Ingham's article [14] treated the Möbius function as well, and showed that a bound of the form $M_0(x)/\sqrt{x} < C$, or $M_0(x)/\sqrt{x} > -C$, for some fixed positive constant C and all sufficiently large x, would also imply the existence of infinitely many integer linear relations among the ordinates of the zeros of the zeta function in the upper half plane. In this case, the function corresponding to $F_0(s)$ from (8) is $F(s) = 1/(s+1/2)\zeta(s+1/2)$. This has no pole at the origin, and the residue at $i\gamma$, where $\rho = 1/2 + i\gamma$ is a zero of $\zeta(s)$, is $\mathrm{Res}(F, i\gamma) = 1/\rho\zeta'(\rho)$. The lack of pole at the origin for $F(s)$ produces qualitatively different behavior for the Möbius sums, compared to the functions $L_\alpha(x)$: the sums $M_\alpha(x)$ exhibit no sign bias.

We remark that Odlyzko and te Riele's disproof of the Mertens conjecture employed the analog of Theorem 4 for $M_0(x)$. They showed that the sum

$$B^*(u) = 2\Re \sum_{0<\gamma_n<T} \frac{k_T(\gamma_n) e^{i\gamma_n u}}{\rho_n \zeta'(\rho_n)}$$

is larger than 1 infinitely often, and smaller than -1 infinitely often. They showed this by constructing values for u so that the most significant terms in this sum lined up nearly in the direction of the positive real axis, or nearly in the direction of the negative real axis. Their method employed the LLL lattice reduction algorithm [20] to solve a simultaneous approximation problem using the 70 most significant zeros among the first 2000 in the upper half plane.

In 2015, Best and the second author [4] also employed the LLL algorithm, but in a different fashion, to improve bounds on oscillations in the Mertens problem. They proved that $M_0(x)/\sqrt{x} > 1.6383$ infinitely often, and that $M_0(x)/\sqrt{x} < -1.6383$ infinitely often. To describe this method, we must first discuss the notion of *weak independence* of a set of real numbers, and an approach to the oscillation problem that employs this idea.

4 Weak Independence

We return to our study of the normalized Liouville sums $A_\alpha(u)$. From Ingham's method, it is known that arbitrarily large oscillations for $A_\alpha(u)$, for each $\alpha \in [0, 1]$, would follow from the linear independence of the ordinates of the zeros of the zeta function in the upper half plane. Beginning in the early 1970s, a number of researchers sought to quantify connections between oscillations in sums of arithmetic functions and the linear independence problem. A key idea is the notion of *weak independence* of a set of real numbers: a set is N-independent if no linear relations besides the trivial one exist involving its members when the integer coefficients of the relations cannot exceed N in absolute value.

Since true independence implies infinite oscillations, it is perhaps not surprising that weak independence might be employed to establish large, though finite, oscillations. Conversely, a bound on the oscillations should perhaps lead to the existence of linear relations among the zeros that employ only small integer coefficients. This latter point of view was developed by Bateman et al. in 1971 [3]. They proved that the Mertens conjecture, which of course asserts a very strong bound on the oscillations of $M_0(x)$, implies the existence of infinitely many linear relations among the zeros of the ordinates of the zeta function in the upper half plane in which only the coefficients ± 1 and 2 appear, with 2 appearing at most once per relation. Diamond [7], Grosswald [10], and Anderson and Stark [2] develop this notion in a more general context. Here, we essentially follow the treatment of Anderson and Stark, as it allows some additional freedom in its application.

Let Γ denote a set of positive real numbers, and let Γ' denote a subset of $\Gamma \cap [0, T]$, for a real parameter T. We say Γ' is *N-independent in $\Gamma \cap [0, T]$* if both of the following conditions hold.

(i) If $\sum_{\gamma \in \Gamma'} c_\gamma \gamma = 0$ with each $c_\gamma \in \mathbb{Z}$ and each $|c_\gamma| \le N$, then each $c_\gamma = 0$.
(ii) If $\sum_{\gamma \in \Gamma'} c_\gamma \gamma = \gamma^*$ for some $\gamma^* \in \Gamma \cap [0, T]$ with each $c_\gamma \in \mathbb{Z}$ and each $|c_\gamma| \le N$, then $\gamma^* \in \Gamma'$, $c_{\gamma^*} = 1$, and every other $c_\gamma = 0$.

That is, no nontrivial integer relations among the elements of Γ' with bounded coefficients exist, and one cannot manufacture an element of Γ with height at most T by using an integer relation among the elements of Γ' with bounded coefficients.

Suppose $g(u)$ is a piecewise continuous real function which is bounded on finite intervals, and let $G(s)$ denote its Laplace transform,

$$G(s) = \int_0^\infty g(u) e^{-su}\, du.$$

Suppose $G(s)$ is absolutely convergent in the half plane $\sigma > \sigma_0$, and can be analytically continued back to $\sigma \ge 0$, except for simple poles on the imaginary axis occurring at $\pm i\gamma$ for $\gamma \in \Gamma$, where Γ is a set of positive real numbers $\Gamma = \{\gamma_n\}$, and possibly at 0 as well. Anderson and Stark established essentially the following theorem.

Theorem 6 *Let $\Gamma' \subset \Gamma$, and suppose Γ' is N-independent in $\Gamma \cap [0,T]$. Then, using the notation above,*

$$\liminf_{u\to\infty} g(u) \le \mathrm{Res}(G,0) - \frac{2N}{N+1} \sum_{\gamma\in\Gamma'} k_T(\gamma)\,|\mathrm{Res}(G,i\gamma)| \tag{13}$$

and

$$\limsup_{u\to\infty} g(u) \ge \mathrm{Res}(G,0) + \frac{2N}{N+1} \sum_{\gamma\in\Gamma'} k_T(\gamma)\,|\mathrm{Res}(G,i\gamma)|\,, \tag{14}$$

where $k_T(t)$ is an admissible weight function.

Here, we recall that an admissible weight function is nonnegative, even, is supported on $[-T,T]$, takes the value 1 at $t=0$, and is the Fourier transform of a nonnegative function. We include the proof of this theorem from [2] for the reader's convenience.

Proof For each $\gamma \in \Gamma$, let $\theta_\gamma = \arg(\mathrm{Res}(G,i\gamma))$ and let $a_\gamma = |\mathrm{Res}(G,i\gamma)|\,k_T(\gamma)$. As in (11), define

$$B_T^*(u) = \mathrm{Res}(G,0) + \sum_{\gamma\in\Gamma\cap[0,T]} a_\gamma \left(\exp(i(\gamma u+\theta_\gamma)) + \exp(-i(\gamma u+\theta_\gamma))\right).$$

For a positive real number U, certainly

$$\frac{1}{2U}\int_{-U}^{U} B_T^*(u) \prod_{\gamma\in\Gamma'} \left(\exp\left(\frac{i(\gamma u+\theta_\gamma)}{2}\right) + \exp\left(-\frac{i(\gamma u+\theta_\gamma)}{2}\right)\right)^{2N} du$$

$$\le \left(\sup_{U\in\mathbb{R}^+} B_T^*(U)\right) \frac{1}{2U}\int_{-U}^{U} \prod_{\gamma\in\Gamma'} \left(\exp\left(\frac{i(\gamma u+\theta_\gamma)}{2}\right) + \exp\left(-\frac{i(\gamma u+\theta_\gamma)}{2}\right)\right)^{2N} du.$$

Expanding the binomial in each term of the product on the right produces the constant term $\binom{2N}{N}$ in each factor. Expanding the product then produces a trigonometric polynomial where the argument of each exponential function is a linear combination of the elements of Γ' where the coefficients are bounded by N. Hence, by the first property of N-independence for Γ', the constant term of this polynomial is $\binom{2N}{N}^{|\Gamma'|}$. The limiting value of the right side as $U\to\infty$ is therefore $\binom{2N}{N}^{|\Gamma'|}$. Expanding the left side and taking the limit, certainly the constant term of $B_T^*(u)$ produces $\mathrm{Res}(G,0)\binom{2N}{N}$. Terms involving $\gamma \in \Gamma\backslash\Gamma'$ from $B_T^*(u)$ contribute nothing, since there is no constant term here, due to the second property of N-independence of Γ' in $\Gamma\cap[0,T]$. Finally, each of the terms involving a $\gamma\in\Gamma'$ from the sum in $B_T^*(u)$ contributes $\left(a_\gamma\binom{2N}{N-1} + a_\gamma\binom{2N}{N+1}\right)\binom{2N}{N}^{|\Gamma'|-1}$ to this limit. Therefore,

$$\mathrm{Res}(G,0)\binom{2N}{N}^{|\Gamma'|} + 2\binom{2N}{N+1}\binom{2N}{N}^{|\Gamma'|-1} \sum_{\gamma\in\Gamma'} a_\gamma \le \binom{2N}{N}^{|\Gamma'|} \sup_{U\in\mathbb{R}^+} B_T^*(U),$$

so

$$\sup_{U\in\mathbb{R}^+} B_T^*(U) \geq \mathrm{Res}(G,0) + \frac{2N}{N+1}\sum_{\gamma\in\Gamma'} k_T(\gamma)\,|\mathrm{Res}(G,i\gamma)|\,.$$

The inequality (14) then follows from an analogue of Theorem 4 for the function $g(u)$, and the statement (13) then follows by replacing $g(u)$ by $-g(u)$ in this argument.

We remark that Anderson and Stark prove a more general statement, in which each coefficient c_γ in a posited linear relation is governed by its own bound of the form $|c_\gamma| \leq N_\gamma$. The conclusion in this more general statement is quite similar—the rational number $N/(N+1)$ in inequalities (13) and (14) is moved inside each respective sum and replaced by $N_\gamma/(N_\gamma+1)$.

5 Lattices and Improved Oscillation Bounds

Theorem 6 and its relatives show that in principle an algorithm exists for establishing large oscillations, since verifying that no linear relations with bounded coefficients exist among a finite set of real numbers is certainly a finite computation. However, such a strategy is computationally prohibitive in cases of interest, even if a modest bound on N is chosen. Recently, Best and the second author [4] determined a way to employ this result of Anderson and Stark in a computationally tractable way, and used it to establish their improved bounds in the Mertens problem cited in Sect. 3. The main idea is to create a number of integer lattices using a set Γ' of selected zeros of the zeta function. These lattices have the property that if Γ' is N-dependent in $\Gamma\cap[0,T]$, then one of them must contain a short nonzero vector. They then establish that none of these lattices can possess such a vector, and apply Theorem 6 using a computed value of N. We apply this method to establish bounds on oscillations for the functions $A_\alpha(u)$, including the improved bounds listed in Theorem 1. Our algorithmic strategy differs from that of [4] in some significant ways that we also detail below.

Let Γ denote the set of ordinates of the zeros of the zeta function in the upper half plane. Fix a value for $\alpha\in[0,1]$. Let T be a positive real parameter, and label the elements of $\Gamma\cap[0,T]$ as $\gamma_1<\gamma_2<\cdots<\gamma_m$. Next, we must select an admissible weight function $k_T(t)$. Recall that this function must be even and supported on $[-T,T]$, satisfy $k_T(0)=1$, and must be the Fourier transform of a nonnegative symmetric function $K_T(y)$. Let n be an integer parameter, with $0<n<m$, and let $\Gamma'=\Gamma'(\alpha,n,T,k_T)$ denote the subset of size n of $\Gamma\cap[0,T]$ consisting of those values of this set for which the quantity

$$k_T(\gamma)\,|\mathrm{Res}(F_\alpha,i\gamma)| = k_T(\gamma)\left|\frac{\zeta(2\rho)}{(\rho-\alpha)\zeta'(\rho)}\right|$$

is largest. Label the elements of Γ' by $\gamma_{a_1} < \cdots < \gamma_{a_n}$. Next, let b be another integer parameter, which governs the precision of our calculations. We construct $m-n+1$ lattices, labeled $\Lambda_0, \ldots, \Lambda_{m-n}$. The lattice Λ_0 has dimension n and lies in $\mathbb{R}^{n+1}$, with a basis given by the column vectors of the $(n+1) \times n$ dimensional matrix formed by appending the row $(\lfloor 2^b \gamma_{a_1} \rceil, \ldots, \lfloor 2^b \gamma_{a_n} \rceil)$ to the bottom of the identity matrix I_n, where $\lfloor x \rceil$ denotes the integer nearest x. For $1 \le i \le m-n$, the lattice Λ_i is constructed in a similar way, by grafting $(\lfloor 2^b \gamma_{a_1} \rceil, \ldots, \lfloor 2^b \gamma_{a_n} \rceil, \lfloor 2^b \gamma^* \rceil)$ onto I_{n+1}, where γ^* is a particular value in $(\Gamma \cap [0,T]) \backslash \gamma'$.

Suppose that Γ' were N-dependent in $\Gamma \cap [0,T]$. Then one of the following conditions would have to hold:

(i) $\sum_{\gamma \in \Gamma'} c_\gamma \gamma = 0$, each $|c_\gamma| \le N$, not all $c_\gamma = 0$.
(ii) $\sum_{\gamma \in \Gamma'} c_\gamma \gamma = \gamma^*$, each $|c_\gamma| \le N$, $\gamma^* \in \Gamma'$, and it is not the case that $c_{\gamma^*} = 1$ and every other $c_\gamma = 0$.
(iii) $\sum_{\gamma \in \Gamma'} c_\gamma \gamma = \gamma^*$, each $|c_\gamma| \le N$, $\gamma^* \in (\Gamma \cap [0,T]) \backslash \Gamma'$.

If case (i) were to occur, then $v = (c_1, c_2, \ldots, c_n, \sum_{i=1}^n c_i \lfloor 2^b \gamma_{a_i} \rceil)$ is a nonzero vector in Λ_0 (writing c_i for $c_{\gamma_{a_i}}$), and we compute

$$\begin{aligned} |v|^2 &= \sum_{i=1}^n c_i^2 + \left(\sum_{i=1}^n c_i \left(\lfloor 2^b \gamma_{a_i} \rceil - 2^b \gamma_{a_i} \right) \right)^2 \\ &\le nN^2 + \left(\sum_{i=1}^n \frac{c_i}{2} \right)^2 \le \left(\frac{n^2}{4} + n \right) N^2. \end{aligned} \tag{15}$$

In the same way, case (ii) would imply the existence of $v \neq 0$ in Λ_0 satisfying

$$|v|^2 \le \left(\frac{n^2}{4} + n \right) N^2 + \left(\frac{n}{2} + 2 \right) N + \frac{5}{4}, \tag{16}$$

and case (iii) would allow one to deduce the existence of $v \neq 0$ in a lattice Λ_i, for some positive index i, where

$$|v|^2 \le \left(\frac{n^2}{4} + n \right) N^2 + \frac{nN}{2} + \frac{5}{4}. \tag{17}$$

It is enough then to show that none of these lattices contains a short nonzero vector. In particular, we need to show that Λ_0 has no nonzero vector satisfying (16), and no Λ_i with $1 \le i \le m-n$ has a nontrivial vector satisfying (17). To establish this, we employ the following property of real lattices. For the reader's convenience we include its proof, adapted from [6, p. 51].

Lemma 1 *Suppose $\{v_1, \ldots, v_n\}$ are linearly independent real vectors, let Λ denote the $\mathbb{Z}$-lattice spanned by this set, and let $\{w_1, \ldots, w_n\}$ denote its Gram–Schmidt orthogonalization. If v is a nonzero vector in Λ, then $|v| \ge \min\{|w_1|, \ldots, |w_n|\}$.*

Proof Suppose v is a nontrivial vector in Λ, and write $v = \sum_{i=1}^{n} a_i v_i$ where each $a_i \in \mathbb{Z}$. Let k denote the largest index where $a_k \neq 0$ in this sum. By the Gram–Schmidt process, we can represent each vector v_i as a linear combination of $w_1, \ldots, w_i$,

$$v = \sum_{i=1}^{k} a_i \sum_{j=1}^{i} r_{ij} w_j = \sum_{j=1}^{k} \left(\sum_{i=j}^{k} a_i r_{ij} \right) w_j = a_k r_{kk} w_k + \sum_{j=1}^{k-1} \left(\sum_{i=j}^{k} a_i r_{ij} \right) w_j,$$

where each $r_{ij} \in \mathbb{R}$. Since $r_{kk} = 1$ owing to the Gram–Schmidt process, and because $w_1, \ldots, w_k$ are orthogonal, we obtain

$$|v|^2 = a_k^2 |w_k|^2 + \sum_{j=1}^{k-1} \left(\sum_{i=j}^{k} a_i r_{ij} \right)^2 |w_j|^2,$$

and we conclude that $|v|^2 \geq |w_k|^2 \geq \min\{|w_1|^2, \ldots, |w_n|^2\}$.

Using Lemma 1 on the bases we constructed for our lattices $\Lambda_0, \ldots, \Lambda_{m-n}$ would produce extremely poor lower bounds on the size of a nontrivial vector in these lattices, since the given basis vectors of these lattices all point roughly in the same direction. However, we are free to apply this lemma to *any* basis of the lattice, and we will certainly obtain much better results if the basis we employ is in some sense more orthogonal. We can create such a basis by preprocessing our basis with the LLL lattice reduction algorithm [20]. Given a basis for an n-dimensional lattice Λ, this algorithm produces a *reduced basis* $\{v_1, \ldots, v_n\}$ having the property that

$$\Delta(\Lambda) \leq \prod_{i=1}^{n} |v_i| \leq \left(\frac{4}{4\delta - 1} \right)^{n(n-1)/4} \Delta(\Lambda),$$

where δ is a free parameter in $(1/4, 1)$. The lower bound is simply Hadamard's inequality; the meaningful part here is the upper bound, which can be interpreted as a guarantee that the basis vectors produced are in a sense more orthogonal. The quality of the basis, as well as the running time of the algorithm, depends on the value of δ: larger values produce a better basis at the expense of longer computation times. The algorithm is guaranteed to run in polynomial time provided δ is selected in this range.

6 Computations

We must select a number of parameters in order to apply this method to the functions $A_\alpha(u)$. First, we select $\alpha \in [0, 1]$, but noting that $|\mathrm{Res}(F_\alpha, i\gamma)| = |\mathrm{Res}(F_{1-\alpha}, i\gamma)|$, we need only consider $\alpha \in [0, 1/2]$. Next, we must select a weight function

$k_T(t)$. Anderson and Stark, following Ingham, employed the Fejér kernel, where $k_T(t) = 1 - |t|/T$ for $t \in [-T, T]$, but after some experimentation we find that the kernel introduced by Jurkat and Peyerimhoff [16] and Odlyzko (see [26]) produces better results. This kernel, which was also employed in the disproof of the Mertens conjecture [25] as well as the recent improvements in this problem [4], is defined by

$$k_T(t) = \begin{cases} (1 - |t|/T)\cos(\pi t/T) + \sin(\pi |t|/T)/\pi, & |t| \le T \\ 0, & |t| > T. \end{cases} \tag{18}$$

Our experiments with the choice of kernel consisted of evaluating the limiting values of the bounds in (13) and (14) as $N \to \infty$ using these two possibilities for $k_T(t)$, for a number of sets Γ' obtained from various choices of T and n. We found that the function (18) consistently produced better results, just as Best and the second author concluded in [4, Fig. 2]. It is possible that another choice of kernel could produce better bounds, but Odlyzko and te Riele [25, Sect. 4.1] indicate that potential improvements from such a change are rather limited.

With our kernel selected, we have four additional parameters to choose for our computation:

- T, the height on the critical line for the zeros of $\zeta(s)$, or equivalently m, the number of zeros in $\Gamma \cap [0, T]$.
- n, the size of the set Γ', and the dimension of the lattice Λ_0.
- b, the number of bits of precision for computing the zeros of $\zeta(s)$.
- δ, the parameter governing the quality of the basis produced by LLL, as well as its run time.

In the main computation of [4], the authors selected n to be rather large, at 500, and b very large as well, at nearly 30,000. This makes for a time-consuming lattice reduction step, but they compensated in part for this by selecting a modest value for δ at 0.3, and picking m somewhat small at 2000. This computation employed 1501 lattices, each of dimension 500 or 501, and produced a value of N close to 5000.

We take a different approach in this computation. We wish to show that $A_0(u) > 1$ infinitely often: this can be computed with a number of different choices of the parameters. If we select n to be small, then we need to check many zeros of $\zeta(s)$ before finding enough with sufficiently large residues, or we need to select T large so that the weight function is more generous at the zeros we pick. In any event this strategy produces many lattices to analyze. Choosing smaller values for n would also encourage selecting a larger δ, which would produce a better result. On the other hand, if we pick n to be large, then we will not need as many zeros, so we will have fewer lattices to reduce, but each one will require more time, and we may need to pick δ smaller.

The LLL algorithm has worst-case running time $O(n^6 b^3)$, but often performs better in practice. Through experiments, we found that b needed to grow approximately linearly in n, and some empirical trials using $b = \Theta(n)$ found that the total run time required by running LLL followed by Gram–Schmidt grew approximately as n^4.

Table 1 Timing estimates (in core-days) for establishing that $A_0(u) > 1$ infinitely often

n	m	b	C	Time
240	5000	6500	1.0033	417
250	3700	7000	1.0034	376
260	3000	7500	1.0023	407
270	2600	8000	1.0023	435
280	2400	8500	1.0056	471
290	2200	9000	1.0065	476
300	2000	9500	1.0042	480

1–89, 91–94, 96–98, 100–108, 111, 113–119, 121, 123–130, 132, 134–136, 138, 140–142, 145–147, 149, 151–153, 156–160, 163, 164, 169, 171, 172, 174, 175, 178, 180–182, 186, 187, 189, 192, 193, 197, 199, 204, 205, 208, 210, 212, 213, 215, 216, 222–224, 227, 229, 231, 233, 235, 241, 243, 247, 253–255, 259, 260, 263, 266, 272, 278, 279, 282, 285, 291, 298, 299, 310, 311, 316, 317, 323, 324, 327, 330, 331, 337, 343, 344, 350, 356, 363, 364, 369, 377, 390, 398, 403, 407, 410, 424, 436, 437, 451, 452, 472, 481, 485, 486, 493, 507, 508, 514, 521, 543, 556, 557, 579, 606, 607, 629, 643, 680, 693, 694, 716, 717, 769, 807, 860, 922, 923, 1278.

Fig. 3 Indices of the 250 zeros of $\zeta(s)$ employed in the calculations

Using this empirical value, we could estimate the time required to obtain our desired result in Pólya's problem. Some estimates are recorded in Table 1. For a number of choices of n, this table lists the number of zeros m required to obtain the desired result, then our estimate for the number of bits of precision b required in order to obtain a value for N near 3000, followed by the best possible result C we could obtain (that is, without the attenuating factor of $N/(N+1)$), and finally the estimated time required to complete this calculation, in core-days on a typical workstation. These estimates assumed $\delta = 0.99$. Our results led us to choose $n = 250$. Some additional experiments indicated that we could likely trim our value of b to 6600, so our final choice of parameters was $n = 250$, $m = 3700$ (and $T = \gamma_{3701} - \epsilon$), $b = 6600$, and $\delta = 0.99$.

The indices a_i of the $n = 250$ zeros selected for this computation are shown in Fig. 3. The numerical values of these zeros were computed to high precision using the `mpmath` Python library [15]. This package was also used to compute the values of the zeta function and its derivative that were needed for the calculation of the residues in (10). We constructed the 3451 matrices for the lattices Λ_i, and then used SageMath [28] to run each of these through LLL followed by Gram–Schmidt. After this, inequalities (16) and (17) were used to determine a lower bound N_i on the length of a nonzero vector in the lattice Λ_i.

These computations were performed on two clusters, one housed in the Research Computing Group at Simon Fraser University, and another that is part of the WestGrid computing consortium of western Canada. The smallest N_i reported from these jobs was 2527; the mean value was 3720. With only 62 of these matrices producing a bound below 3100, we checked these cases a second time at a higher precision, choosing $b = 6950$. This raised the value for each of these cases to at least 6702, so we may use $N = 3100$ as our value for weak independence. We therefore obtain the following result.

Theorem 7 *The set* $\Gamma' = \{\gamma_{a_i} : 1 \le i \le 250\}$, *where the sequence* a_i *is shown in Fig. 3, is* 3100*-independent in* $\gamma \cap [0, T]$, *with* $T = \gamma_{3701} - 10^{-10}$.

Using this result in Theorem 6 with $G = F_0$ and $k_T(t)$ given by (18) immediately produces Theorem 1.

We may now use Theorem 7 to deduce bounds on the oscillations of other sums involving the Liouville function. By choosing $G = F_\alpha$ in Theorem 6 with $\alpha = 1/4$, $1/2$, or $3/4$, employing formulas for the residues shown in (9) and (10), using the kernel (18) again, and translating the results for the functions $A_\alpha(u)$ back to the sums $L_\alpha(x)$ by using (5) and (6), we obtain the following theorem.

Theorem 8 *Each of the following inequalities is satisfied for infinitely many integers x.*

$$L_{1/4}(x) < -3.0573x^{1/4}, \; L_{1/4}(x) > 0.31824x^{1/4}, \tag{19}$$

$$L_{1/2}(x) - \frac{\log x}{2\zeta(1/2)} < -1.1634, \; L_{1/2}(x) - \frac{\log x}{2\zeta(1/2)} > 2.2122, \tag{20}$$

$$L_{3/4}(x) - \frac{\zeta(3/2)}{\zeta(3/4)} < -\frac{0.31824}{x^{1/4}}, \; L_{3/4}(x) - \frac{\zeta(3/2)}{\zeta(3/4)} > \frac{3.0573}{x^{1/4}}. \tag{21}$$

It may seem that the inequalities in Theorem 8 could be improved by tuning our calculation for the specific cases $\alpha = 1/2$ or $\alpha = 3/4$. After all, the residues in (10) exhibit a dependence on α, so it seems possible that a different set Γ' would be optimal for these choices. However, with the same choices for n and m, we find that the optimal set of indices for Γ' for these cases is precisely the list shown in Fig. 3. Thus, the results of Theorem 8 cannot be improved with another calculation of the same size by using this method.

The same method allows us to establish Theorem 2. Figure 4 shows the lower bound on $\limsup_{u\to\infty} A_\alpha(u)$, as a function of α, that we obtain from Theorem 6 by using the independence result of Theorem 7. This graph has a root when $1/\zeta(1/2)(1-2\alpha) = 1.6878471\ldots$, so when $\alpha = 0.297148\ldots$, and this establishes the first statement in the theorem. The statement that $\liminf_{u\to\infty} A_\alpha(u)$ is infinitely often positive for $\alpha > 0.702851\ldots$ then follows by symmetry.

As α is taken to be closer to $1/2$ it is harder to exhibit an infinitude of sign changes. This is not surprising since then the term $\mathrm{Res}(F_\alpha, 0)$ in (9) is larger and hence exerts greater dominance in (11). Indeed Humphries [13, Theorems 1.5, 1.6] has proved some conditional results about the biases in $L_\alpha(x)$ when α tends to $1/2$.

7 Improvements

We close by briefly noting some prospects for further improvements in these oscillation bounds. As noted after the proof of Theorem 6, one could choose a bound on c_i that depends on i. If, for example, $|c_i| \le N(1 - i/N)$, then the

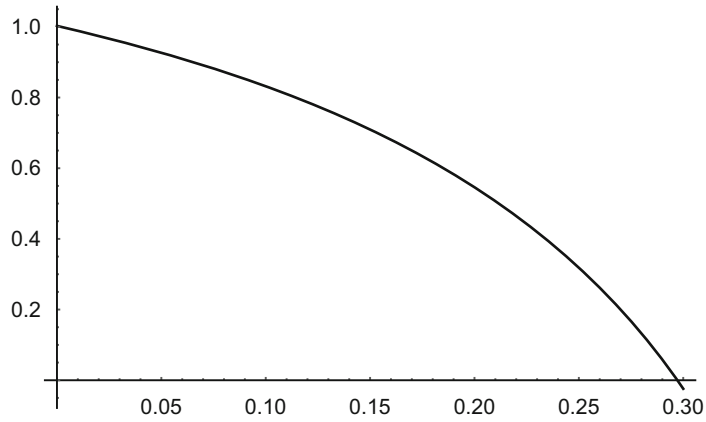

Fig. 4 Lower bound for $\limsup_{u\to\infty} A_\alpha(u)$, for $0 \le \alpha \le 0.3$

sums $(\sum_{i=1}^{n} c_i/2)^2$ in (15)–(17) would have a main term of $(Nn)^2/16$ instead of $(Nn)^2/4$. This is a win; we now must balance this against the loss of attenuating the contribution from large zeros. Although these contributions are small, and hence we expect an overall win with this strategy, we have not pursued this further.

We note also the qualitative gains that come from using the weak linear independence method. Prior to this method, one exhibited large oscillations in a function like $L_0(x)$ by searching for a single large value $L_0(x_0)$—cf. (4). To find an even larger oscillation, one needs to search for a larger spike in $L_0(x)$. There is no guarantee that such a spike will be found by extending the computational run time modestly. Indeed, given that $L_0(x)$ also obtains large negative values, one may have to extend the run time considerably to witness the next spike. Using this approach, it is difficult to exhibit large oscillations in $L_0(x)$, because $L_0(x)$ has large oscillations!

In contrast, if one increases the size of the matrices in our method, or accounts for more zeros, then one is *guaranteed* a better bound in the oscillations. However, in the absence of any new idea, we believe such improvements will be small without radically larger computation times. To illustrate the marginal nature of this improvement, we note that Gonek [8] and Hejhal [12] independently conjectured that $\sum_{0<\gamma<T} |\zeta'(\rho)|^{-1} \asymp T(\log T)^{1/4}$. Littlewood [21] showed that $(\log\log t)^{-1} \ll \zeta(1+it) \ll \log\log t$, whence the behavior of $\zeta(2\rho)$ should be fairly benign. Therefore, if the Gonek and Hejhal conjecture were true it would follow from partial summation that

$$\sum_{0<\gamma_n\le N} \left|\frac{\zeta(2\rho_n)}{\rho_n\zeta'(\rho_n)}\right| \asymp (\log N)^{5/4}.$$

Therefore, to increase our sums, and hence to improve on our bounds on the oscillations, we would require substantially many more zeros.

Acknowledgements We thank the Centre for Interdisciplinary Research in the Mathematical and Computational Sciences (IRMACS) at Simon Fraser University, and WestGrid, the Western Canada Research Grid and regional partner of Compute Canada, for computational resources. We also thank the anonymous referee for helpful comments.

Research of M.J. Mossinghoff was partially supported by grants from the Simons Foundation (#210069 and #426694). Research of T.S. Trudgian was partially supported by a Future Fellowship (FT160100094) from the Australian Research Council.

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Explorations in the Theory of Partition Zeta Functions

Ken Ono, Larry Rolen, and Robert Schneider

Abstract We introduce and survey results on two families of zeta functions connected to the multiplicative and additive theories of integer partitions. In the case of the multiplicative theory, we provide specialization formulas and results on the analytic continuations of these "partition zeta functions," find unusual formulas for the Riemann zeta function, prove identities for multiple zeta values, and see that some of the formulas allow for p-adic interpolation. The second family we study was anticipated by Manin and makes use of modular forms, functions which are intimately related to integer partitions by universal polynomial recurrence relations. We survey recent work on these zeta polynomials, including the proof of their Riemann Hypothesis.

1 The Setting: Visions of Euler

In antiquity, storytellers began their narratives by invoking the muse whose divine influence would guide the unfolding imagery. It is fitting, then, that we begin this article by praising the immense curiosity of Euler, whose imagination ranged playfully across almost the entire landscape of modern mathematical thought.

Euler made spectacular use of product–sum relations, often arrived at by unexpected avenues, thereby inventing one of the principle archetypes of modern number theory. Among his many profound identities is the product formula for what is now called the Riemann zeta function:

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} = \prod_{p \in \mathbb{P}} (1 - p^{-s})^{-1} \tag{1}$$

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H. Montgomery et al. (eds.), *Exploring the Riemann Zeta Function*,
DOI 10.1007/978-3-319-59969-4_10

With this relation, Euler connected the (at the time) cutting-edge theory of infinite series to the ancient set $\mathbb{P}$ of prime numbers. Moreover, in solving the famous "Basel problem" posed a century earlier by Pietro Mengoli (1644), Euler showed us how to compute even powers of π using the zeta function, giving explicit formulas of the shape

$$\zeta(2N) = \pi^{2N} \times \text{ rational}. \tag{2}$$

It turns out there are other classes of zeta functions, arising from other Eulerian formulas in the universe of partition theory.

Much like the set of positive integers, but perhaps even more richly, the set of integer partitions ripples with striking patterns and beautiful number-theoretic phenomena. In fact, the positive integers $\mathbb{N}$ are embedded in the integer partitions $\mathscr{P}$ in a number of ways: obviously, positive integers themselves represent the set of partitions into one part; less trivially, the prime decompositions of integers are in bijective correspondence with the set of prime partitions, i.e., the partitions into prime parts (if we map the number 1 to the empty partition $\emptyset$), as Alladi and Erdős note [1]. We might also identify the divisors of n with the partitions of n into identical parts, and there are many other interesting ways to associate integers to the set of partitions.

Euler found another profound product–sum identity, the generating function for the partition function $p(n)$

$$\prod_{n=1}^{\infty}(1-q^n)^{-1} = \sum_{n=0}^{\infty} p(n)q^n, \tag{3}$$

single-handedly establishing the theory of integer partitions. This formula doesn't look much like the zeta function identity (1); however, generalizing Euler's proofs of these theorems leads to a new class of partition-theoretic zeta functions. It turns out that (1) and (3) both arise as specializations of a single "master" product–sum formula.

Before we proceed, let us fix some notation. Let $\mathscr{P}$ denote the set of all integer partitions. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$, denote a generic partition, $l(\lambda) := r$ denote its *length* (the number of parts), and $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_r$ denote its *size* (the number being partitioned). We write "$\lambda \vdash n$" to mean λ is a partition of n, and "$\lambda_i \in \lambda$" to indicate $\lambda_i \in \mathbb{N}$ is one of the parts of λ.

Then we have the following "master" identity (which is a piece of Theorem 1.1 in [41]).

Proposition 1 *For an arbitrary function $f : \mathbb{N} \to \mathbb{C}$, we have*

$$\prod_{n=1}^{\infty}(1-f(n)q^n)^{-1} = \sum_{\lambda\in\mathscr{P}} q^{|\lambda|} \prod_{\lambda_i\in\lambda} f(\lambda_i).$$

The sum on the right is taken over all partitions, and the left-hand product is taken over the parts λ_i of partition λ. The proof proceeds along similar lines to Euler's proof of (3), and can be seen immediately if one expands a few terms of the infinite product by hand, without collecting coefficients in the usual manner. This simple identity also has interesting (and sometimes exotic) representations in terms of Eulerian q-series and continued fractions; readers are referred to [41] for details.

Remark 1 Note that, collecting like terms, we can rewrite such partition sums as standard power series, summed over non-negative integers:

$$\sum_{\lambda \in \mathscr{P}} c_\lambda q^{|\lambda|} = \sum_{n=0}^{\infty} q^n \sum_{\lambda \vdash n} c_\lambda \tag{4}$$

It is immediate from Proposition 1 and (4) that the partition generating function (3) results if we let $f \equiv 1$ identically.

Similarly, if we set $f(n) = n^{-s}, q = 1$, and sum instead over the subset $\mathscr{P}_{\mathbb{P}}$ of prime partitions, we arrive at the Euler product formula (1) for the zeta function. This follows from the bijective correspondence between prime partitions and the factorizations of natural numbers noted above.

In light of these observations, we might view the Riemann zeta function as the prototype for a new class of combinatorial objects arising from Eulerian methods.

1.1 Partition-Theoretic Zeta Functions

Inspired by work of Euler [19], Fine [23], Andrews [2], Bloch and Okounkov [7], Zagier [53], Alladi and Erdős [1], and others, the authors here undertake the study of a class of zeta functions introduced by the third author in [41], resembling the Riemann zeta function $\zeta(s)$ but summed over proper subsets of $\mathscr{P}$, as opposed to over natural numbers.

In this paper, we review a few of the results from [41], and record a number of further identities relating certain zeta functions arising from the theory of partitions to various objects in number theory such as Riemann zeta values, multiple zeta values, and infinite product formulas. Some of these formulas are related to results in the literature; they are presented here as examples of this new class of partition-theoretic zeta functions. We also give several formulas for the Riemann zeta function, and results on the analytic continuation (or non-existence thereof) of zeta-type series formed in this way. Furthermore, we discuss the p-adic interpolation of these zeta functions in analogy with the classical work of Kubota and Leopoldt on p-adic continuation of the Riemann zeta function [33].

To describe our primary object of study, we must introduce a new statistic related to partitions, to complement the length and size defined above. We define the *integer* of a partition λ, notated as n_λ, to be the product of its parts:

$$n_\lambda := \lambda_1 \lambda_2 \dots \lambda_r \tag{5}$$

This multiplicative statistic may not seem very natural as partitions arise purely additively, with deep additive structures such as Ramanujan congruences dominating the theory. Yet if we let n_λ formally replace the usual index n in the Riemann zeta function, and sum instead over appropriate subsets of partitions, we arrive at an analytic-combinatorial object with many nice properties.

Definition 1 Over a subset $\mathscr{P}' \subset \mathscr{P}$ and value $s \in \mathbb{C}$ for which the sum converges, we define a *partition zeta function* to be the series

$$\zeta_{\mathscr{P}'}(s) := \sum_{\lambda \in \mathscr{P}'} n_\lambda^{-s}.$$

Very nice relations arise from unique properties of special subsets $\mathscr{P}'$, e.g., partitions with some distinguishing structure, or into parts sharing some trait. For example, if we let $\mathscr{P}^* \subset \mathscr{P}$ denote partitions into distinct parts, there are interesting closed-form expressions, such as

$$\zeta_{\mathscr{P}^*}(2) = \frac{\sinh \pi}{\pi}, \tag{6}$$

which follows from Proposition 1 together with a formula of Euler (cf. [41]). By summing instead over partitions into parts ≥ 2 (that is, disallowing 1s), we arrive at curious identities such as

$$\zeta_{\mathscr{P}_{\geq 2}}(3) = \frac{3\pi}{\cosh\left(\frac{1}{2}\pi\sqrt{3}\right)}, \tag{7}$$

which can be found from Proposition 1 together with a formula of Ramanujan [41]. This equation is somewhat surprising, as the Riemann zeta values at odd arguments are famously enigmatic.

Other attractive closed sums can be found—and general structures observed, as detailed in [41]—when we restrict our attention to partitions $\mathscr{P}_{\mathscr{M}}$ whose parts all lie in a subset $\mathscr{M} \subset \mathbb{N}$. It is easy to check from Proposition 1 that we have the Euler product formula

$$\zeta_{\mathscr{P}_{\mathscr{M}}}(s) = \prod_{k \in \mathscr{M}} (1 - k^{-s})^{-1}. \tag{8}$$

We see from the right-hand side of (8) that $\zeta_{\mathscr{P}_{\mathscr{M}}}(s)$ diverges if $1 \in \mathscr{M}$, thus the restriction $1 \notin \mathscr{M}$ exhibited in the pair of identities above is a necessary one here. In fact, some restriction on the maximum multiplicity of 1 occurring as a part is necessary for any partition zeta function to converge. That is, we must sum over partitions containing 1 with multiplicity at most $m \geq 0$. Note that the resulting zeta function will equal the one in the case where 1s are not allowed, multiplied by $m+1$, as each n_λ^{-s} is repeated $m+1$ times in the sum (adjoining 1s to a partition does not affect its "integer").

We note that if Dirichlet series coefficients a_n are defined by

$$\zeta_{\mathscr{P}_{\mathscr{M}}}(s) =: \sum_{n\geq 1} a_n n^{-s},$$

it is easy to see that a_n counts the number of ways to write n as a product of integers in $\mathscr{M}$, where each ordering of factors is only counted once. When $\mathscr{M} = \mathbb{N}$, then these ways of writing n as a product of smaller numbers are known as *multiplicative partitions*, and have been studied in a number of places in the literature; for example, the interested reader is referred to [2, 13, 34, 43, 54].

We wish to study partition zeta functions over special subsets of $\mathscr{P}$ and arguments s that lead to interesting relations. We begin by highlighting a few nice-looking examples. Let us recall a few identities from [41] as examples of zeta function phenomena induced by suitable partition subsets of the form $\mathscr{P}_{\mathscr{M}}$. By summing over partitions into even parts we have a combinatorial formula to compute π:

$$\zeta_{\mathscr{P}_{2\mathbb{N}}}(2) = \frac{\pi}{2} \tag{9}$$

Curiously, this partition zeta function produces an *odd* power of π, which does not occur in Euler's evaluations of ζ at even arguments. This simple formula belongs to an infinite family of identities arrived at by letting $z = \pi/m$ in Euler's product identity for the sine function

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right). \tag{10}$$

To describe this family, we take $\mathscr{P}_{m\mathbb{N}} \subsetneq \mathscr{P}$ to denote the set of partitions into multiples of $m > 1$. Summing over these partitions, we have identities such as

$$\zeta_{\mathscr{P}_{m\mathbb{N}}}(2) = \frac{\pi}{m \sin\left(\frac{\pi}{m}\right)}, \tag{11}$$

$$\zeta_{\mathscr{P}_{m\mathbb{N}}}(4) = \frac{\pi^2}{m^2 \sin\left(\frac{\pi}{m}\right) \sinh\left(\frac{\pi}{m}\right)}, \tag{12}$$

and similar (but increasingly complicated) formulas can be computed for $\zeta_{\mathscr{P}_{m\mathbb{N}}}(2^t)$ with $t \in \mathbb{N}$. Clearly (9) is the case $m = 2$ of the first identity.

All of these partition zeta formulas seem to hint at analogous algebraicity results like that of (2) for the Riemann zeta function. Although the first few examples show that the situation is more complicated, such formulas immediately lead one to suspect the existence of a similar theory.

There is indeed a class of partition zeta functions with a structure much like that exhibited by classical zeta functions.

Definition 2 We define the partition zeta function $\zeta_{\mathscr{P}}(\{s\}^k)$ to be the following sum over all partitions λ of fixed length $\ell(\lambda) = k \geq 0$:

$$\zeta_{\mathscr{P}}(\{s\}^k) := \sum_{\ell(\lambda)=k} \frac{1}{n_\lambda^s} \quad (\mathrm{Re}(s) > 1)$$

We directly find that $\zeta_{\mathscr{P}}(\{s\}^0) = n_\emptyset^{-s} = 1$ and $\zeta_{\mathscr{P}}(\{s\}^1) = \zeta(s)$. Proceeding much as Euler did to evaluate $\zeta(2k)$ (cf. [19]), for $k \geq 0$ we find that $\zeta_{\mathscr{P}}(\{2\}^k)$ is a rational multiple of $\zeta(2k)$:

$$\zeta_{\mathscr{P}}(\{2\}^k) = \frac{2^{2k-1}-1}{2^{2k-2}}\zeta(2k) \tag{13}$$

It follows that $\zeta_{\mathscr{P}}(\{2\}^k)$ is of the form "$\pi^{2k}\times$rational" for all positive k, much like the zeta values $\zeta(2k)$ given by Euler. Equation (13) also suggests the well-known Riemann zeta value

$$\zeta(0) = \frac{2^{-2}}{2^{-1}-1}\zeta_{\mathscr{P}}(\{2\}^0) = -1/2,$$

which is usually arrived at via analytic continuation. Furthermore, we can write $\zeta_{\mathscr{P}}(\{2^t\}^k)$ explicitly for all $t \in \mathbb{N}$ as finite combinations of well-known zeta values, also of the shape "$\pi^{2^t k} \times$ rational" [41].

Of course, it was Riemann's creative analytic continuation of the zeta function that led to stunning advances in number theory and analysis. Using general structural relations for partition zeta functions, the analytic continuation of $\zeta_{\mathscr{P}}(\{s\}^k)$ is given in [41] for one case: for fixed length $k = 2$, we can write

$$\zeta_{\mathscr{P}}(\{s\}^2) = \frac{\zeta(2s)+\zeta(s)^2}{2}. \tag{14}$$

Thus $\zeta_{\mathscr{P}}(\{s\}^2)$ inherits analytic continuation from the two zeta functions on the right. One would very much like to see further examples of analytic continuations of $\zeta_{\mathscr{P}}(\{s\}^k)$ as well as other partition zeta functions.

We remark that (14) resembles the well-known "series shuffle product" relation for multiple zeta values (see [4]). Hoffman gives lovely formulas using different notations, relating $\zeta_{\mathscr{P}}(\{s\}^k)$ to combinations of multiple zeta values [27, 28]. As we will see below, the theory of partition zeta functions connects with multiple zeta values in other ways as well.

1.2 Evaluations

In the previous section we saw a variety of simple closed forms for partition zeta functions, depending on the natures of the subsets of partitions being summed over. Different subsets induce different zeta phenomena. In what follows, we consider the evaluations of a small sampling of possible partition zeta functions having particularly pleasing formulas.

1.2.1 Zeta Functions for Partitions with Parts Restricted by Congruence Conditions

Our first line of study will concern sets $\mathscr{M} \subset \mathbb{N}$ that are defined by congruence conditions. Note by (8) that for disjoint sets $\mathscr{M}_1$ and $\mathscr{M}_2$,

$$\zeta_{\mathscr{P}_{\mathscr{M}_1 \cup \mathscr{M}_2}}(s) = \zeta_{\mathscr{P}_{\mathscr{M}_1}}(s)\zeta_{\mathscr{P}_{\mathscr{M}_2}}(s).$$

Hence, to study any set of partitions determined by congruence conditions on the parts, it suffices to consider series of the form

$$\zeta_{\mathscr{P}_{a+m\mathbb{N}}}(s),$$

where $a \in \mathbb{Z}_{\geq 0}$, $m \in \mathbb{N}$ (excluding the case $a = 0$, $m = 1$, where the zeta function clearly diverges), and $\mathscr{P}_{a+m\mathbb{N}}$ is partitions into parts congruent to a modulo m. We see examples of the case $\zeta_{\mathscr{P}_{m\mathbb{N}}}(2^N) = \zeta_{\mathscr{P}_{0+m\mathbb{N}}}(2^N)$ in (11) and (12); we are interested in the most general case, with $s = n \in \mathbb{N}$.

Our first main result is then the following, where Γ is the usual gamma function of Euler and $e(x) := e^{2\pi i x}$. The proof will use an elegant and useful formula highlighted by Chamberland and Straub in [12], which we note was also inspired by previous work on multiplicative partitions in [13]. In fact, the following result is a generalization of Theorem 8 of [13] which in our notation corresponds to $a = m = 1$.

Theorem 1 *For $n \geq 2$, we have*

$$\zeta_{\mathscr{P}_{a+m\mathbb{N}}}(n) = \Gamma(1 + a/m)^{-n} \prod_{r=0}^{n-1} \Gamma\left(1 + \frac{a - e(r/n)}{m}\right).$$

Theorem 1 has several applications. By considering the expansion of the logarithm of the gamma function, we easily obtain the following result, in which γ is the Euler–Mascheroni constant and the principal branch of the logarithm is taken.

Corollary 1 *For any $m, n \geq 2$, we have that*

$$\log\left(\zeta_{\mathscr{P}_{a+m\mathbb{N}}}(n)\right) = n\log(1+a/m) + \frac{a(n+1)}{m}(1-\gamma) - \sum_{r=0}^{n-1}\log\left(1+\frac{a-e(r/n)}{m}\right)$$
$$+\sum_{r=0}^{n-1}\sum_{k\geq 2}\frac{(-1)^k(\zeta(k)-1)\left(a^k+(a-e(r/n))^k\right)}{km^k}.$$

When $a = 0$ and $m \geq 2$, we obtain the following strikingly simple formula, which is similar to Theorem 7 of [13] that in our notation corresponds to the case $a = m = 1$.

Corollary 2 *For any $m, n \geq 2$, we have that*

$$\log\left(\zeta_{\mathscr{P}_{m\mathbb{N}}}(n)\right) = n\sum_{\substack{k\geq 2\\ n|k}}\frac{\zeta(k)}{km^k}.$$

1.2.2 Connections to Ordinary Riemann Zeta Values

In addition to providing interesting formulas for values of more exotic partition-theoretic zeta functions, the above results also lead to curious formulas for the classical Riemann zeta function. In fact, $\zeta(s)$ is itself a partition zeta function, summed over prime partitions, so it is perhaps not too surprising to find that we can learn something about it from a partition-theoretic perspective. Then we continue the theme of evaluations by recording a few results expressing the value of ζ at integer argument $n > 1$ in terms of gamma factors.

In the first, curious identity, let μ denote the classical Möbius function. We point out that this is essentially a generalization of a formula for the case $a = m = 1$ given in Eq. (11) of [13].

Corollary 3 *For all $m, n \geq 2$, we have*

$$\zeta(n) = m^n\sum_{k\geq 1}\frac{\mu(k)}{k}\sum_{r=0}^{nk-1}\log\left(\Gamma\left(1-\frac{e\left(\frac{r}{nk}\right)}{m}\right)\right).$$

The next identity gives $\zeta(n)$ in terms of the nth derivative of a product of gamma functions. The authors were not able to find this formula in the literature; however, given the well-known connections between Γ and ζ, as well as the known example below the following theorem, it is possible that the identity is known.

Theorem 2 *For integers $n > 1$, we have*

$$\zeta(n) = \frac{1}{n!}\lim_{z\to 0^+}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\prod_{j=0}^{n-1}\Gamma\left(1-ze(j/n)\right).$$

Example 1 As an example of implementing the above identity, take $n = 2$; then using Euler's well-known product formula for the sine function, it is easy to check that

$$\zeta(2) = \frac{1}{2!}\lim_{z\to 0^+}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Gamma(1+z)\,\Gamma(1-z) = \frac{1}{2!}\lim_{z\to 0^+}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\frac{\pi z}{\sin(\pi z)} = \frac{\pi^2}{6}.$$

This last formula for $\zeta(n)$, following from a formula in [41] together with the preceding theorem, is analogous to some extent to the classical identity $\sin(n) = \frac{e^{in}-e^{-in}}{2i}$.

Corollary 4 *For integers $n > 1$, we have*

$$\zeta(n) = \lim_{z\to 0^+}\frac{\prod_{j=0}^{n-1}\Gamma(1-ze(j/n)) - \prod_{j=0}^{n-1}\Gamma(1-ze(j/n))^{-1}}{2z^n}.$$

1.2.3 Zeta Functions for Partitions of Fixed Length

We now consider zeta sums of the shape $\zeta_{\mathscr{P}}(\{s\}^k)$ as in Definition 2. Our first aim will be to extend (13) above (which is Corollary 2.4 of [41]). As we shall see below, these zeta values are special cases of sums considered by Hoffman in [27].

Let $[z^n]f$ represent the coefficient of z^n in a power series f. Using this notation, we show the following, which in particular gives an algorithmic way to compute each $\zeta_{\mathscr{P}}(\{m\}^k)$ in terms of Riemann zeta values for $m \in \mathbb{N}_{\geq 2}$.

Theorem 3 *For all $m \geq 2$, $k \in \mathbb{N}$, we have*

$$\begin{aligned}\zeta_{\mathscr{P}}(\{m\}^k) &= \pi^{mk}[z^{mk}]\prod_{r=0}^{m-1}\Gamma\left(1-\frac{z}{\pi}e(r/m)\right)\\ &= \pi^{mk}[z^{mk}]\exp\left(\sum_{j\geq 1}\frac{\zeta(mj)}{j}\left(\frac{z}{\pi}\right)^{mj}\right).\end{aligned}$$

Generalizing the comments just below (13), the next corollary follows directly from Theorem 3 (using the well-known fact that $\zeta(k) \in \mathbb{Q}\pi^k$ for even integers k).

Corollary 5 *For $m \in 2\mathbb{N}$ even, we have that*

$$\zeta_{\mathscr{P}}(\{m\}^k) \in \mathbb{Q}\pi^{mk}.$$

Remark 2 This can also be deduced from Theorem 2.1 of [27].

To conclude this section, we note one explicit method for computing the values $\zeta_{\mathscr{P}}(\{m\}^k)$ at integral k, m (especially if m is even, in which case the zeta values below are completely elementary).

Corollary 6 *For $m \geq 2, k \in \mathbb{N}$, and $j \geq i$, set*

$$\alpha_{i,j} := \zeta(m(j-i+1))\frac{(k-i)!}{\pi^{m(j-i+1)}(k-j)!}.$$

Then we have

$$\zeta_{\mathscr{P}}(\{m\}^k) = \frac{\pi^{mk}}{k!}\det\begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \dots & \alpha_{1,k} \\ -1 & \alpha_{2,2} & \alpha_{2,3} & \dots & \alpha_{2,k} \\ 0 & -1 & \alpha_{3,3} & \dots & \alpha_{3,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 & \alpha_{k,k} \end{pmatrix}.$$

Remark 3 There are results resembling these in Knopfmacher and Mays [31].

1.3 Analytic Continuation and p-Adic Continuity

If we jump forward about 100 years from the pathbreaking work of Euler concerning special values of the Riemann zeta function at even integers, we arrive at the famous work of Riemann in connection with prime number theory. Namely, in 1859, Riemann brilliantly described the most significant properties of $\zeta(s)$ following that of an Euler product: the analytic continuation and functional equation for $\zeta(s)$. It is for this reason, of course, that the zeta function is named after Riemann, and not Euler, who had studied this function in some detail, and even conjectured a related functional equation. In particular, this analytic continuation allowed Riemann to bring the zeta function, and indeed the relatively new field of complex analysis, to the forefront of number theory by connecting its roots to the distribution of prime numbers.

It is natural therefore, whenever one is faced with new zeta functions, to ask about their prospect for analytic continuation. Here, we offer a brief study of some of these properties, in particular showing that the situation for our zeta functions is much more singular. Partition-theoretic zeta functions in fact naturally give rise to functions with essential singularities. Here, we use Corollary 2 to study the continuation properties of partition zeta functions over partitions $\mathscr{P}_{m\mathbb{N}}$ into multiples of $m > 1$. In order to state the result we first define, for any $\varepsilon > 0$, the right half-plane $\mathbb{H}_\varepsilon := \{z \in \mathbb{C} : \mathrm{Re}(z) > \varepsilon\}$, and we denote by $\frac{1}{\mathbb{N}}$ the set $\{1/n : n \in \mathbb{N}\}$.

Corollary 7 *For any $\varepsilon > 0$ and $m > 1$, $\zeta_{\mathscr{P}_{m\mathbb{N}}}(s)$ has a meromorphic extension to $\mathbb{H}_\varepsilon$ with poles exactly at $\mathbb{H}_\varepsilon \cap \frac{1}{\mathbb{N}}$. In particular, there is no analytic continuation beyond the right half-plane $\mathrm{Re}(s) > 0$, as there would be an essential singularity at $s = 0$.*

Remark 4 For the function $\zeta_{\mathscr{P}_{\mathbb{N}}}(s)$, a related discussion of poles and analytic continuation was made by the user mohammad-83 in a mathoverflow.net question.

Finally, we follow Kubota and Leopoldt [33], who showed ζ could be modified slightly to obtain modified zeta functions for any prime p which extend ζ to the space of p-adic integers $\mathbb{Z}_p$, to obtain further examples of p-adic zeta functions of this sort. These continuations are based on the original observations of Kubota and Leopoldt, and, in a rather pleasant manner, on the evaluation formulas discussed above.

In particular, we will use Corollary 6 to p-adically interpolate modified versions of $\zeta_{\mathscr{P}}(\{m\}^k)$ in the m-aspect. Given the connection discussed in Sect. 1.4 to multiple zeta values, these results should be compared with the literature on p-adic multiple zeta values (e.g., see [25]), although we note that our p-adic interpolation procedure seems to be more direct in the special case we consider.

The continuation in the m-aspect of this function is also quite natural, as the case $k = 1$ is just that of the Riemann zeta function. Thus, it is natural to search for a suitable p-adic zeta function that specializes to the function of Kubota and Leopoldt when $k = 1$. It is also desirable to find a p-adic interpolation result which makes the partition-theoretic perspective clear.

Here, we provide such an interpretation. Let us first denote the set of partitions with parts not divisible by p as $\mathscr{P}_p$; then we consider the length-k partition zeta values $\zeta_{\mathscr{P}_p}(\{s\}^k)$. Note that for $k = 1$, $\zeta_{\mathscr{P}_p}(\{s\}^1)$ is just the Riemann zeta function with the Euler factor at p removed (as considered by Kubota and Leopoldt). We then offer the following p-adic interpolation result.

Theorem 4 *Let $k \geq 1$ be fixed, and let $p \geq k + 3$ be a prime. Then $\zeta_{\mathscr{P}_p}(\{s\}^k)$ can be extended to a continuous function for $s \in \mathbb{Z}_p$ which agrees with $\zeta_{\mathscr{P}_p}(\{s\}^k)$ on a positive proportion of integers.*

1.4 Connections to Multiple Zeta Values

Our final application of the circle of ideas related to partition zeta functions and infinite products will be in the theory of multiple zeta values.

Definition 3 We define for natural numbers $m_1, m_2, \ldots, m_k$ with $m_k > 2$ the *multiple zeta value* (commonly written "MZV")

$$\zeta(m_1, m_2, \ldots, m_k) := \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{m_1} \ldots n_k^{m_k}}.$$

We call k the *length* of the MZV. Furthermore, if $m_1 = m_2 = \cdots = m_k$ are all equal to some $m \in \mathbb{N}$, we use the common notation

$$\zeta(\{m\}^k) := \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{(n_1 n_2 \ldots n_k)^m}. \tag{15}$$

Multiple zeta values have a rich history and enjoy widespread connections; interested readers are referred to Zagier's short note [51], and for a more detailed treatment, the excellent lecture notes of Borwein and Zudilin [9]. There are many nice closed-form identities in the literature; for example, one can show (see [27]) on analogy to (13) that

$$\zeta(\{2\}^k) = \frac{\pi^{2k}}{(2k+1)!}. \tag{16}$$

Similar (but more complicated) expressions for $\zeta(\{2^t\}^k)$ for all $t \in \mathbb{N}$ are given in [41], parallel to those mentioned in Sect. 1.1 for $\zeta_{\mathscr{P}}(\{2^t\}^k)$.

Observe that the partition zeta function $\zeta_{\mathscr{P}}(\{m\}^k)$ (see Definition 13) can be rewritten in a similar-looking form to (15) above:

$$\zeta_{\mathscr{P}}(\{m\}^k) = \sum_{n_1 \geq n_2 \geq \cdots \geq n_k \geq 1} \frac{1}{(n_1 n_2 \ldots n_k)^m} \tag{17}$$

In fact, if we take $\mathscr{P}^*$ to denote partitions into distinct parts, then (15) reveals $\zeta(\{m\}^k)$ is equal to the partition zeta function $\zeta_{\mathscr{P}^*}(\{m\}^k)$ summed over length-k partitions into distinct parts, as pointed out in [41]. Series such as those in (17) have been considered and studied extensively by Hoffman (for instance, see [27]).

By reorganizing sums of the shape (17), we arrive at interesting relations between $\zeta_{\mathscr{P}}(\{m\}^k)$ and families of MZVs. In order to describe these relations, we first recall that a *composition* is simply a finite tuple of natural numbers, and we call the sum of these integers the *size* of the composition. Denote the set of all compositions by $\mathscr{C}$ and write $|\lambda| = k$ for $\lambda = (a_1, a_2, \ldots, a_j) \in \mathscr{C}$ if $k = a_1 + a_2 + \cdots + a_j$. Then we obtain the following.

Proposition 2 *Assuming the notation above, we have that*

$$\zeta_{\mathscr{P}}(\{m\}^k) = \sum_{\substack{\lambda=(a_1,\ldots,a_j)\in\mathscr{C} \\ |\lambda|=k}} \zeta(a_1 m, a_2 m, \ldots, a_j m).$$

Remark 5 Proposition 2 is analogous to results of Hoffman; the reader is referred to Theorem 2.1 of [27].

In particular, for any $n > 1$ we can find the following reduction of $\zeta(\{n\}^k)$ to MZVs of smaller length. We note in passing that Theorem 2.1 of [27] also shows directly how to write these values in terms of products (as opposed to simply linear combinations) of ordinary Riemann zeta values: hints, perhaps, of further connections. We remark in passing that this can be thought of as a sort of "parity result" (cf. [29, 45]).

Corollary 8 *For any $n, k > 1$, the MZV $\zeta(\{n\}^k)$ of length k can be written as an explicit linear combination of MZVs of lengths less than k.*

As our final result, we give a simple formula for $\zeta(\{n\}^k)$. This formula is probably already known; if $k = 2$ it follows from a well-known result of Euler (see the discussion of $H(n)$ on p. 3 of [52]) and is closely related to (11) and (32) of [8]. The idea of the proof is also similar to what has appeared in, for example, [52]. However, the authors have decided to include it due to connections with the ideas used throughout this paper, and the simple deduction of the formula from expressions necessary for the proofs of the results described above.

Proposition 3 *The MZV $\zeta(\{n\}^k)$ of length k can be expressed as a linear combination of products of ordinary ζ values. In particular, we have*

$$\zeta(\{n\}^k) = (-1)^k \left[z^{nk}\right] \exp\left(-\sum_{j\geq 1} \frac{\zeta(nj)}{j} z^{nj}\right).$$

Remark 6 This formula is equivalent to a special case of Theorem 2.1 of [27]. However, since the proof is very simple and ties in with the other ideas in this paper, we give a proof for the reader's convenience.

The proof of Corollary 6 yields a similar determinant formula here.

Corollary 9 *For $n \geq 2, k \in \mathbb{N}$, and $j \geq i$, set*

$$\beta_{i,j} := -\zeta(n(j-i+1))\frac{(k-i)!}{(k-j)!}.$$

Then we have

$$\zeta(\{n\}^k) = \frac{(-1)^k}{k!} \det \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & \beta_{1,3} & \dots & \beta_{1,k} \\ -1 & \beta_{2,2} & \beta_{2,3} & \dots & \beta_{2,k} \\ 0 & -1 & \beta_{3,3} & \dots & \beta_{3,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 & \beta_{k,k} \end{pmatrix}.$$

Remark 7 We can see from the above corollary that $\zeta(\{n\}^k)$ is a linear combination of products of zeta values, which is closely related to formulas of Hoffman [27].

1.5 Machinery

1.5.1 Useful Formulas

In this section, we collect several formulas that will be key to the proofs of the theorems above. We begin with the following beautiful formula given by Chamberland and Straub in Theorem 1.1 of [12]. In fact, this formula has a long history, going back at least to Sect. 12.13 of [47], and we note that Ding, Feng, and Liu independently discovered this same result in Lemma 7 of [17].

Theorem 5 *If $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are complex numbers, none of which are non-positive integers, with $\sum_{j=1}^{n} \alpha_j = \sum_{j=1}^{n} \beta_j$, then we have*

$$\prod_{k\geq 0} \prod_{j=1}^{n} \frac{(k+\alpha_j)}{(k+\beta_j)} = \prod_{j=1}^{n} \frac{\Gamma(\beta_j)}{\Gamma(\alpha_j)}.$$

We will also require two Taylor series expansions for $\log \Gamma$, both of which follow easily from Euler's product definition of the gamma function [20]. The first expansion, known as Legendre's series, is valid for $|z| < 1$ (see (17) of [48]):

$$\log \Gamma(1+z) = -\gamma z + \sum_{k\geq 2} \frac{\zeta(k)}{k}(-z)^k. \tag{18}$$

We also have the following expansion valid for $|z| < 2$ (see (5.7.3) of [36]):

$$\log \Gamma(1+z) = -\log(1+z) + z(1-\gamma) + \sum_{k\geq 2} (-1)^k (\zeta(k)-1)\frac{z^k}{k}. \tag{19}$$

Furthermore, we need a couple of facts about Bell polynomials (see Chap. 12.3 of [2]). The nth *complete Bell polynomial* is the sum

$$B_n(x_1, \ldots, x_n) := \sum_{i=1}^{n} B_{n,i}(x_1, x_2, \ldots, x_{n-i+1}).$$

The ith term here is the polynomial

$$B_{n,i}(x_1, x_2, \ldots, x_{n-i+1}) := \sum \frac{n!}{j_1! j_2! \cdots j_{n-i+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-i+1}}{(n-i+1)!}\right)^{j_{n-i+1}},$$

where we sum over all sequences $j_1, j_2, \ldots, j_{n-i+1}$ of non-negative integers such that $j_1 + j_2 + \cdots + j_{n-i+1} = i$ and $j_1 + 2j_2 + 3j_3 + \cdots + (n-i+1)j_{n-i+1} = n$.

With these notations, we use a specialization of the classical Faà di Bruno formula [22], which allows us to write the exponential of a formal power series as a power series with coefficients related to complete Bell polynomials:

$$\exp\left(\sum_{j=1}^{\infty} \frac{a_j}{j!} x^j\right) = \sum_{k=0}^{\infty} \frac{B_k(a_1, \ldots, a_k)}{k!} x^k \tag{20}$$

Faà di Bruno also gives an identity [22] that specializes to the following formula for the kth complete Bell polynomial in the series above as the determinant of a certain $k \times k$ matrix:

$$B_k(a_1,\ldots,a_k) = \det \begin{pmatrix} a_1 & \binom{k-1}{1}a_2 & \binom{k-1}{2}a_3 & \binom{k-1}{3}a_4 & \ldots & \ldots & a_k \\ -1 & a_1 & \binom{k-2}{1}a_2 & \binom{k-2}{2}a_3 & \ldots & \ldots & a_{k-1} \\ 0 & -1 & a_1 & \binom{k-3}{1}a_2 & \ldots & \ldots & a_{k-2} \\ 0 & 0 & -1 & a_1 & \ldots & \ldots & a_{k-3} \\ 0 & 0 & 0 & -1 & \ldots & \ldots & a_{k-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -1 & a_1 \end{pmatrix} \tag{21}$$

1.5.2 Proofs of Theorems 1 and 2, and Their Corollaries

We begin with the proof of our first main formula.

Proof (Proof of Theorem 1) By (8), we find that

$$\zeta_{\mathscr{P}_{a+m\mathbb{N}}}(n) = \prod_{k\in a+m\mathbb{N}} \frac{k^n}{k^n-1} = \prod_{j\geq 1} \frac{(a+mj)^n}{(a+mj)^n-1} = \prod_{j\geq 0}\prod_{r=0}^{n-1} \frac{(j+1+a/m)^n}{\left(j+1+\frac{a-e(r/n)}{m}\right)}.$$

Using Theorem 5 and the well-known fact that

$$\sum_{j=0}^{n-1} e(j/n) = 0 \tag{22}$$

directly gives the desired result.

Proof (Proof of Corollary 1) For this, we apply (19) and use (22), the obvious fact that

$$|(a-e(j/n))/m| < 2,$$

and the easily checked fact that

$$1+(a-e(j/n))$$

is never a negative real number for $j = 0,\ldots,n-1$.

Proof (Proof of Corollary 2) Here, we simply use (18). Again, the corollary is proved following a short, elementary computation, using the classical fact that

$$\sum_{r=0}^{n-1} e(rk/n) = \begin{cases} n & \text{if } n|k, \\ 0 & \text{else.} \end{cases}$$

Proof (Proof of Corollary 7) By Corollary 2, we find for $n \geq 2$ that

$$\log\left(\zeta_{\mathscr{P}_{m\mathbb{N}}}(n)\right) = \sum_{k\geq 1} \frac{\zeta(nk)}{km^{kn}}.$$

Suppose that $\mathrm{Re}(s) > 0$ and $s \notin \frac{1}{\mathbb{N}}$. Then letting

$$K := \max\{\lceil 1/\mathrm{Re}(s)\rceil + 1, \mathrm{Re}(s)\},$$

it clearly suffices to show that

$$\sum_{k\geq K} \frac{\zeta(sk)}{km^{ks}}$$

converges. But in this range on k, by choice we have $\mathrm{Re}(sk) > 1$, so that using the assumption $m \geq 2$, we find for $\mathrm{Re}(s) > 0$ the upper bound

$$\begin{aligned}\sum_{k\geq K} \frac{\zeta(sk)}{km^{ks}} &\leq \zeta(Ks) \sum_{k\geq K} \frac{1}{k2^{k\,\mathrm{Re}(s)}} \leq \zeta(Ks) \sum_{k\geq 1} \frac{1}{k2^{k\,\mathrm{Re}(s)}} \\ &= -\zeta(Ks) \log\left(2^{-\mathrm{Re}(s)}\left(2^{\mathrm{Re}(s)} - 1\right)\right),\end{aligned}$$

and note that in the argument of the logarithm in the last step, by assumption we have $2^{\mathrm{Re}(s)} - 1 > 0$.

Conversely, if $s \in \frac{1}{\mathbb{N}}$, then it is clear that this representation shows there is a pole of the extended partition zeta function, as one of the terms gives a multiple of $\zeta(1)$.

Proof (Proof of Corollary 3) We utilize a variant of Möbius inversion, reversing the order of summation in the double sum $\sum_{k\geq 1}\sum_{d|k} \mu(d) f(nk) k^{-s}$; if

$$g(n) = \sum_{k\geq 1} \frac{f(kn)}{k^s},$$

then

$$f(n) = \sum_{k\geq 1} \frac{\mu(k) g(kn)}{k^s}.$$

Applying this inversion procedure to Corollary 2, so that $g(n) = \log \zeta_{\mathscr{P}_{m\mathbb{N}}}(n)$ (taking $s = 1$), and $f(n) = \zeta(n)/m^n$, we directly find that

$$\zeta(n) = m^n \sum_{k\geq 1} \frac{\mu(k)}{k} \log\left(\zeta_{\mathscr{P}_{m\mathbb{N}}}(nk)\right).$$

Applying Theorem 1 then gives the result.

Proof (Proof of Theorem 2) By the comments following Theorem 1.1 in [41], for $\mathscr{M} \in \mathbb{N}$ we have

$$\prod_{k\in\mathscr{M}} \left(1 - \frac{z^s}{k^s}\right)^{-1} = 1 + z^s \sum_{k\in\mathscr{M}} \frac{1}{k^s \prod_{\substack{j\in\mathscr{M}\\ j\leq k}} \left(1 - \frac{z^s}{j^s}\right)}; \tag{23}$$

thus

$$\sum_{k\in\mathscr{M}} k^{-s} = \lim_{z\to 0^+} \frac{\prod_{k\in\mathscr{M}} \left(1-\frac{z^s}{k^s}\right)^{-1} - 1}{z^s}.$$

Taking $\mathscr{M} = \mathbb{N}, s = n \in \mathbb{Z}_{\geq 2}$, we apply L'Hospital's rule n times to evaluate the limit on the right-hand side. The theorem then follows by noting, from Theorem 5, that in fact

$$\prod_{k\in\mathbb{N}} \left(1-\frac{z^n}{k^n}\right)^{-1} = \prod_{j=0}^{n-1} \Gamma\left(1-ze(j/n)\right).$$

Proof (Proof of Corollary 4) Picking up from the proof of Theorem 2 above, it follows also from Theorem 1.1 of [41] that

$$\prod_{k\in\mathscr{M}} \left(1-\frac{z^s}{k^s}\right) = 1 - z^s \sum_{k\in\mathscr{M}} \frac{\prod_{\substack{j\in\mathscr{M}\\ j<k}} \left(1-\frac{z^s}{j^s}\right)}{k^s}.$$

Subtracting this equation from (23), making the substitutions $\mathscr{M} = \mathbb{N}, s = n \geq 2$ as in the proof above, and using Theorem 5, gives the corollary.

1.5.3 Proof of Theorem 3 and Its Corollaries

Proof (Proof of Theorem 3) Using a similar method as in [41] and a similar rewriting to that used in the proof of Theorem 1, we note that a short elementary computation shows

$$\sum_{k\geq 0} \frac{z^{mk}}{\pi^{mk}} \zeta_{\mathscr{P}}(\{m\}^k) = \prod_{k\geq 1} \frac{1}{1-\frac{z^m}{\pi^m k^m}} = \prod_{k\geq 0} \prod_{r=0}^{m-1} \frac{(k+1)^m}{\left(k+1-\frac{z}{\pi}e(r/m)\right)}.$$

Much as in the proof of Theorem 2, using Theorem 5, we directly find that this is equal to

$$\prod_{r=0}^{m-1} \Gamma\left(1-\frac{z}{\pi}e(r/m)\right),$$

which gives the first equality in the theorem. Applying Eq. (18) (formally we require $|z| < \pi$, but we are only interested in formal power series here anyway), we find immediately, using a very similar calculation to that in the proof of Corollary 2, that

$$\sum_{k\geq 0}\left(\frac{z}{\pi}\right)^{mk}\zeta_{\mathscr{P}}(\{m\}^k)=\exp\left(\sum_{r=0}^{m-1}\sum_{j\geq 2}\frac{\zeta(j)}{j}\left(\frac{z}{\pi}\right)^{mj}e(rj/m)\right)$$
$$=\exp\left(m\sum_{\substack{j\geq 2\\ m|j}}\frac{\zeta(j)}{j}\left(\frac{z}{\pi}\right)^{j}\right), \tag{24}$$

which is equivalent to the second equality in the theorem.

Proof (Proof of Corollary 6) Replace x with z^m in Eq. (20), and set

$$a_j=\frac{(j-1)!\zeta(mj)}{\pi^{mj}}$$

on the left-hand side [which becomes the right-hand side of (24)]. Then comparing the right side of (20) to the left side of (24), we deduce that

$$\zeta_{\mathscr{P}}(\{m\}^k)=\frac{\pi^{mk}}{k!}B_k(a_1,\ldots,a_k).$$

To complete the proof, we substitute the determinant in (21) for $B_k(a_1,\ldots,a_k)$ and rewrite the terms in the upper half of the resulting matrix as $\alpha_{i,j}$, as defined in the statement of the corollary.

Proof (Proof of Theorem 4) In analogy with the calculation of Theorem 3, we find that

$$\sum_{k\geq 0}z^{mk}\zeta_{\mathscr{P}_p}(\{m\}^k)=\prod_{\substack{k\geq 1\\ p\nmid k}}\frac{1}{1-\frac{z^m}{k^m}}$$
$$=\frac{\prod_{k\geq 0}\prod_{r=0}^{m-1}\frac{(k+1)^m}{(k+1-ze(r/m))}}{\prod_{k\geq 0}\prod_{r=0}^{m-1}\frac{(k+1)^m}{\left(k+1-\frac{z}{p}e(r/m)\right)}}=\prod_{r=0}^{m-1}\frac{\Gamma\left(1-ze(r/m)\right)}{\Gamma\left(1-\frac{z}{p}e(r/m)\right)}.$$

As in the calculation of (24), this is equal to

$$\exp\left(\sum_{j\geq 1}\frac{\zeta(mj)}{j}(z)^{mj}\left(1-1/p^{mj}\right)\right),$$

so if we set

$$\alpha_{i,j}^{(p)}(m):=\zeta^*(m(j-i+1))\frac{(k-i)!}{(k-j)!},$$

where

$$\zeta^*(s) := (1-p^{-s})\zeta(s),$$

then we have

$$\zeta_{\mathscr{P}_p}(\{m\}^k) = \frac{1}{k!}\det\begin{pmatrix} \alpha_{1,1}^{(p)} & \alpha_{1,2}^{(p)} & \alpha_{1,3}^{(p)} & \dots & \alpha_{1,k}^{(p)} \\ -1 & \alpha_{2,2}^{(p)} & \alpha_{2,3}^{(p)} & \dots & \alpha_{2,k}^{(p)} \\ 0 & -1 & \alpha_{3,3}^{(p)} & \dots & \alpha_{3,k}^{(p)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 & \alpha_{k,k}^{(p)} \end{pmatrix}.$$

We further define $\zeta_{\mathscr{P}_p}(\{m\}^k)$ for more general values in $\mathbb{C}$, such as $m \in -\mathbb{N}$ using the analytic continuation of ζ in each of the factors $\alpha_{i,j}^{(p)}(m)$. Next we recall the Kummer congruences, which state that if k_1, k_2 are positive even integers not divisible by $(p-1)$ and $k_1 \equiv k_2 \pmod{p^{a+1}-p^a}$ for $a \in \mathbb{N}$ where $p > 2$ is prime, then

$$\left(1-p^{k_1-1}\right)\frac{B_{k_1}}{k_1} \equiv \left(1-p^{k_2-1}\right)\frac{B_{k_2}}{k_2} \pmod{p^{a+1}}.$$

Let us take S_{s_0} to be the set of natural numbers congruent to s_0 modulo $p-1$. The Kummer congruences then imply that for any $s_0 \not\equiv 0 \pmod{p-1}$, and for any $k_1, k_2 \in S_{s_0}$ with $k_1 \equiv k_2 \pmod{p^a}$ and $k_1, k_2 > 1$, that

$$\zeta^*(1-k_1) \equiv \zeta^*(1-k_2) \pmod{p^{a+1}}.$$

If we choose $m_1, m_2 \in S_{s_0}$ with $m_1 \equiv m_2 \pmod{p^a}$, then the values $1-(1-m_1)(j-i+1)$, $1-(1-m_2)(j-i+1)$ are in $S_{1+(s_0-1)(j-i-1)}$ and are congruent modulo p^a, and as $p > k$ the additional factorial terms (inside and outside the determinant) are p-integral. Now in our determinant, $j-i+1$ ranges through $\{1, 2, \dots, k\}$, and we want to find an s_0 such that $1+(s_0-1)r \not\equiv 0 \pmod{p-1}$ for $r \in \{1, 2, \dots k\}$. If we take $s_0 = 2$, then the largest value of $1+(s_0-1)r$ is $k+1$, which is by assumption less than $p-1$, and hence not divisible by it. Hence, in our case, $s_0 = 2$ suffices. Thus, if $m_1, m_2 \in S_2$ with $m_1 \equiv m_2 \pmod{p^a}$, then

$$\zeta_{\mathscr{P}_p}(\{1-m_1\}^k) \equiv \zeta_{\mathscr{P}_p}(\{1-m_2\}^k) \pmod{p^{a+1}}.$$

This shows that our zeta function is uniformly continuous on S_2 in the p-adic topology. As this set is dense in $\mathbb{Z}_p$, we have shown that the function extends in the m-aspect to $\mathbb{Z}_p$.

1.5.4 Proofs of Results Concerning Multiple Zeta Values

Proof (Proof of Proposition 2) Recall from (17) that we need to study the sum

$$\sum_{n_1 \geq n_2 \geq \cdots \geq n_k \geq 1} \frac{1}{(n_1 n_2 \dots n_k)^m}.$$

The proof is essentially combinatorial accounting, keeping track of the number of ways to split up a sum

$$\sum_{n_1 \geq n_2 \geq \cdots \geq n_k \geq 1}$$

over all k-tuples of natural numbers into a chain of equalities and strict inequalities. Suppose that we have

$$n_1 \geq n_2 \geq \cdots \geq n_k \geq 1.$$

Then if any of these inequalities is an equality, say $n_j = n_{j+1}$, in the contribution to the sum

$$\sum_{n_1 \geq n_2 \geq \cdots \geq n_k \geq 1} (n_1 \dots n_k)^{-m},$$

the terms n_j and n_{j+1} "double up." That is, we can delete the n_{j+1} and replace the n_j^{-m} in the sum with an n_j^{-2m}. Thus, the reader will find that our goal is to keep track of different orderings of $>$ and $=$, taking symmetries into account. The possible chains of $=$ and $>$ are encoded by the set of compositions of size k, by associating to the composition $(a_1, \dots, a_j)$ the chain of inequalities

$$n_1 = \cdots = n_{a_1} > n_{a_1+1} = \cdots = n_{a_1+a_2} > n_{a_2+1} > \cdots > n_k.$$

That is, the number a_1 determines the number of initial terms on the right which are equal before the first inequality, a_2 counts the number of equalities in the next block of inequalities, and so on. It is clear that the sum corresponding to the each composition then contributes the desired amount to the partition zeta value in the corollary.

Proof (Proof of Corollary 8) In Proposition 2, comparison with Corollary 6 shows that we have a linear relation among MZVs and products of zeta values. Observe that in $\zeta_{\mathscr{P}}(\{m\}^k)$, the only composition of length k is $(1, 1, \dots, 1)$, which contributes $k!\zeta(\{m\}^k)$ to the right-hand side of Proposition 2, and that the rest of the compositions are of lower length, hence giving MZVs of smaller length; the corollary follows immediately.

Proof (Proof of Proposition 3) Consider the multiple zeta value $\zeta(\{n\}^k)$ of length k. Then we directly compute

$$\sum_{k\geq 0}(-1)^k\zeta(\{n\}^k)z^{nk} = \prod_{m\geq 1}\left(1-\left(\frac{z}{m}\right)^n\right) = \prod_{m\geq 0}\prod_{r=0}^{n-1}\frac{(m+1-ze(r/n))}{(m+1)^n}.$$

By Theorem 5, this equals

$$\prod_{r=0}^{n-1}\Gamma(1-ze(r/n))^{-1}.$$

Using precisely the same computation as was made in the proof of Theorem 3, we find that this is equal to

$$\exp\left(-n\sum_{\substack{j\geq 2\\ n|j}}\frac{\zeta(j)}{j}z^j\right).$$

Hence, we have that

$$\zeta(\{n\}^k) = (-1)^k\left[z^{nk}\right]\exp\left(-\sum_{j\geq 1}\frac{\zeta(nj)}{j}z^{nj}\right).$$

Proof (Proof of Corollary 9) Here we proceed exactly as in the proof of Corollary 6, except we make the simpler substitution

$$a_k = (k-1)!\zeta(nk)$$

into Eq. (20), and compare with Proposition 3. In the final step, we replace the terms in the upper half of the matrix with $\beta_{i,j}$ as defined in the statement of the corollary.

1.6 Some Further Thoughts

We have presented samples of a few varieties of flora one finds at the fertile intersection of combinatorics and analysis. What unifies all of these is the perspective that they represent instances of partition zeta functions, with proofs that fit naturally into the Eulerian theory we propound.

We close this article by noting a general class of partition-theoretic analogs of classical Dirichlet series having the form

$$\mathscr{D}_{\mathscr{P}'}(f,s) := \sum_{\lambda \in \mathscr{P}'} f(\lambda) n_\lambda^{-s},$$

where $\mathscr{P}'$ is a proper subset of $\mathscr{P}$ and $f : \mathscr{P}' \to \mathbb{C}$. Of course, partition zeta functions arise from the specialization $f \equiv 1$, just as in the classical case.

Taking $\mathscr{P}' = \mathscr{P}_{\mathscr{M}}$ as defined previously, then if $f := f(n_\lambda)$ is completely multiplicative with appropriate growth conditions, it follows from Theorem 1.1 of [41] that we have a generalization of (8)

$$\mathscr{D}_{P_{\mathscr{M}}}(f,s) = \prod_{j \in \mathscr{M}} \left(1 - \frac{f(j)}{j^s}\right)^{-1} \quad (\mathrm{Re}(s) > 1)\,, \tag{25}$$

and nearly the entire theory of partition zeta functions noted here (and developed in [41]) extends to these series as well.

2 Zeta Polynomials

We now turn our attention toward a different layer of connections between partitions and zeta functions, via the theory of modular forms. Although Euler's generating function for $p(n)$ is essentially modular, and Euler also anticipated the study of L-functions that are intimately tied to modular forms, the true depth of such observations did not come into view until further work on complex analysis was carried out in the nineteenth century.

It turns out that all modular forms are related to partitions in a very direct way. Here we recall the case of this connection for modular forms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. We then use these modular forms to define the second class of functions that are the topic of this paper, the *zeta polynomials* associated with modular forms.

2.1 Partitions and Modular Forms

In a paper from 2004 [11], the first author, Bruinier, and Kohnen investigated the values of a certain sequence of modular functions in connection with the arithmetic properties of meromorphic modular forms on $\mathrm{SL}_2(\mathbb{Z})$. One of the main results in the paper shows that integer partitions and a universal sequence of polynomials encode the Fourier expansions of modular forms.

Here we recall this result, which can be thought of as a precise formulation of the assertion that a modular form is distinguished by its "first few coefficients." In fact, by making use of partitions we have an effective recursive procedure which computes the Fourier coefficients in order.

We take $q := e^{2\pi iz}$ throughout this section. Now, suppose that

$$f(z) = \sum_{n=h}^{\infty} a_f(n) q^n$$

is a weight $k \in 2\mathbb{Z}$ meromorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$. If $k \geq 2$ is even, then let $E_k(z)$ denote the normalized Eisenstein series

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n. \tag{26}$$

Here B_k denotes the usual kth Bernoulli number and $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$. If $k > 2$, then $E_k(z)$ is a weight k modular form on $\mathrm{SL}_2(\mathbb{Z})$; although the Eisenstein series

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n \tag{27}$$

is not a modular form, it also plays an important role. As usual, let $j(z)$ denote the modular function on $\mathrm{SL}_2(\mathbb{Z})$ which is holomorphic on $\mathbb{H}$, the upper half of the complex plane, with Fourier expansion

$$j(z) := q^{-1} + 744 + 196884q + 21493760q^2 + \cdots.$$

We will require a specific sequence of modular functions $j_m(z)$; to define this sequence, we set

$$j_0(z) := 1 \qquad \text{and} \qquad j_1(z) := j(z) - 744. \tag{28}$$

If $m \geq 2$, then define $j_m(z)$ by

$$j_m(z) := j_1(z) \mid T_0(m), \tag{29}$$

where $T_0(m)$ is the usual normalized mth weight zero Hecke operator. Each $j_m(z)$ is a monic polynomial in $j(z)$ of degree m. Here we list the first few:

$$j_0(z) = 1,$$

$$j_1(z) = j(z) - 744 = q^{-1} + 196884q + \cdots,$$

$$j_2(z) = j(z)^2 - 1488j(z) + 159768 = q^{-2} + 42987520q + \cdots,$$

$$j_3(z) = j(z)^3 - 2232j(z)^2 + 1069956j(z) - 36866976 = q^{-3} + 2592899910q + \cdots.$$

Let $\mathfrak{F}$ denote the usual fundamental domain of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. By assumption, $\mathfrak{F}$ does not include the cusp at ∞. Throughout, let $i := \sqrt{-1}$ and let $\omega := (1+\sqrt{-3})/2$. If $\tau \in \mathfrak{F}$, then define e_τ by

$$e_\tau := \begin{cases} 1/2 & \text{if } \tau = i, \\ 1/3 & \text{if } \tau = \omega, \\ 1 & \text{otherwise.} \end{cases} \tag{30}$$

By studying the logarithmic derivatives of modular forms, the first author, Bruinier, and Kohnen obtained the next result in [11], which offers universal polynomial recursion formulas for the Fourier coefficients of modular forms.

Theorem 6 *For every positive integer n define $F_n(x_1, \dots, x_n) \in \mathbb{Q}[x_1, \dots, x_n]$ by*

$$F_n(x_1, \dots, x_n) := -\frac{2x_1\sigma_1(n)}{n} + \sum_{\substack{m_1,\dots,m_{n-1}\geq 0, \\ m_1+2m_2+\cdots+(n-1)m_{n-1}=n}} (-1)^{m_1+\cdots+m_{n-1}} \cdot \frac{(m_1+\cdots+m_{n-1}-1)!}{m_1!\cdots m_{n-1}!} \cdot x_2^{m_1}\cdots x_n^{m_{n-1}}.$$

If $f(z) = q^h + \sum_{n=1}^{\infty} a_f(h+n)q^{h+n}$ is a weight k meromorphic modular form on $SL_2(\mathbb{Z})$, then for every positive integer n we have

$$a_f(h+n) = F_n(k, a_f(h+1), \dots, a_f(h+n-1)) - \frac{1}{n}\sum_{\tau\in\mathfrak{F}} e_\tau \mathrm{ord}_\tau(f)\cdot j_n(\tau).$$

Remark 8 Theorem 6 illustrates a deep connection between integer partitions and the coefficients of modular forms. An inspection of the summation in the theorem reveals the role of partitions: the partitions of n, apart from (n) itself, determine the recurrence formula for $a_f(h+n)$. We note that for each such partition

$$\lambda = ((n-1)^{m_{n-1}}(n-2)^{m_{n-2}}\dots 2^{m_2}1^{m_1}) \vdash n$$

with m_k being the multiplicity of k as a part, we can rewrite $m_1+m_2+\cdots+m_{n-1} = \ell(\lambda)$.

The first few polynomials F_n are

$$F_1(x_1) := -2x_1,$$

$$F_2(x_1, x_2) := -3x_1 + \frac{x_2^2}{2},$$

$$F_3(x_1, x_2, x_3) := -\frac{8x_1}{3} - \frac{x_2^3}{3} + x_2x_3,$$

$$F_4(x_1, x_2, x_3, x_4) := -\frac{7x_1}{2} - x_2^2x_3 + x_2x_4 + \frac{x_2^4}{4} + \frac{x_3^2}{2}.$$

The $n = 1$ case of Theorem 6 implies that

$$a_f(h+1) = 60k - 744h - \sum_{\tau \in \mathfrak{F}} e_\tau \mathrm{ord}_\tau(f) \cdot j(\tau).$$

Example 1 Since

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n = q - 24q^2 + 252q^3 - \cdots,$$

the unique normalized weight 12 cusp form on $\mathrm{SL}_2(\mathbb{Z})$ is nonvanishing in $\mathbb{H}$, Theorem 6 implies that

$$\tau(n+1) = F_n(12, \tau(2), \ldots, \tau(n)).$$

2.2 Manin's Zeta Polynomials

Theorem 6 shows that integer partitions and universal polynomials play central roles in computing the Fourier coefficients of modular forms. The only additional data required is a form's leading coefficient, weight, and its divisor. In the spirit of the previous section, it is then natural to ask for a theory of zeta functions related to partitions, which somehow arises from this connection. Here we address this problem by giving a brief exposition of recent work on a problem of Manin on zeta polynomials.

To begin, we consider any newform $f = \sum_{n \geq 1} a_n q^n \in S_k(\Gamma_0(N))$ of even weight k and level N. The standard zeta-type function associated with f is its L-function

$$L(f, s) := \sum_{n \geq 1} a_n q^n,$$

which may be normalized so that the completed L-function

$$\Lambda(f, s) := \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(f, s)$$

may be analytically continued and satisfies the functional equation

$$\Lambda(f,s) = \varepsilon(f)\Lambda(f,k-s),$$

with $\varepsilon(f) = \pm 1$. The *critical L-values* are the complex numbers $L(f,1)$, $L(f,2)$, $\ldots$, $L(f,k-1)$. Manin's conjecture then states that these critical L-values can be suitably packaged to fit into the following framework.

Definition 4 (Manin) A polynomial $Z(s)$ is a **zeta polynomial** if it satisfies the following criteria:

1. (Naturality) It is arithmetic-geometric in origin.
2. (Functional Equation) For $s \in \mathbb{C}$ we have $Z(s) = \pm Z(1-s)$.
3. (Riemann Hypothesis) If $Z(\rho) = 0$, then $\mathrm{Re}(\rho) = 1/2$.
4. The values $Z(-n)$ have a "nice" generating function
5. The values $Z(-n)$ encode arithmetic-geometric information.

We remark that, of course, it is very important that the arithmetic-geometric nature of the function in part (1) of the definition is satisfied. For example, any polynomial with real coefficients and satisfying the Riemann Hypothesis automatically satisfies the functional equation in part (2), and hence the compelling arithmetic nature of the particular object being constructed is critical. Part (4) is also central to Manin's idea, and is meant to parallel Euler's power series expansion for the Riemann zeta function:

$$\frac{t}{1-e^{-t}} = 1 + \frac{1}{2}t - t\sum_{n=1}^{\infty}\zeta(-n)\cdot\frac{t^n}{n!} \tag{31}$$

This generating function (31) for the values $\zeta(-n)$ also has a well-known interpretation in K-theory [24], in line with part (5) of the above definition. Namely, it is essentially the generation function for the torsion of the K-groups for $\mathbb{Q}$.

Recently in [37], Sprung and the first two authors confirmed Manin's speculations and offered the following resolution to his question. To describe this result, first consider the mth *weighted moments* of critical values:

$$M_f(m) := \sum_{j=0}^{k-2}\left(\frac{\sqrt{N}}{2\pi}\right)^{j+1}\frac{L(f,j+1)}{(k-2-j)!}j^m = \frac{1}{(k-2)!}\sum_{j=0}^{k-2}\binom{k-2}{j}\Lambda(f,j+1)j^m. \tag{32}$$

For positive integers n, we recall the usual generating function for the (signed) Stirling numbers of the first kind:

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1) =: \sum_{m=0}^{n} s(n,m)x^m \tag{33}$$

Using these numbers, the zeta function satisfying Manin's definition is given by

$$Z_f(s) := \sum_{h=0}^{k-2}(-s)^h \sum_{m=0}^{k-2-h} \binom{m+h}{h} \cdot s(k-2, m+h) \cdot M_f(m). \tag{34}$$

The main result of [37] confirms that this natural object satisfies a zeta-type functional equation and the Riemann Hypothesis.

Theorem 7 (Theorem 1.1 of [37]) *If $f \in S_k(\Gamma_0(N))$ is an even weight $k \geq 4$ newform, then the following are true:*

1. *For all $s \in \mathbb{C}$ we have that $Z_f(s) = \varepsilon(f) Z_f(1-s)$.*
2. *If $Z_f(\rho) = 0$, then $\mathrm{Re}(\rho) = 1/2$.*

In accordance with Definition 4, we also want to find natural interpretations of the values $Z_f(-n)$ at negative integers and for their generating functions. This may be accomplished by considering the "Rodriguez-Villegas Transform" of [40]. Theorem 7 is naturally related to the arithmetic of period polynomials[1]

$$R_f(z) := \sum_{j=0}^{k-2} \binom{k-2}{j} \cdot \Lambda(f, k-1-j) \cdot z^j. \tag{35}$$

The values of $Z_f(s)$ at non-positive integers are then the coefficients expanded around $z = 0$ of the rational function

$$\frac{R_f(z)}{(1-z)^{k-1}}.$$

The following result is the analogue of (31) for the Riemann zeta function, and is our answer to part (4) of the above definition.

Theorem 8 (Theorem 1.3 of [37]) *Assuming the notations and hypotheses above, as a power series in z we have*

$$\frac{R_f(z)}{(1-z)^{k-1}} = \sum_{n=0}^{\infty} Z_f(-n) z^n.$$

In accordance with the aforementioned K-theoretic interpretation of the values of $\zeta(s)$ at negative integers, and in relation to part (5) of Definition 4, it is natural to ask whether the z-series in Theorem 8 has an analogous interpretation—that is, what (if any) arithmetic information is encoded by the values $Z_f(-n)$? Hints along these lines were first offered by Manin in [35], where he produced similar

[1] This is a slight renormalization of the period polynomials considered in references such as [15, 32, 38, 50].

zeta polynomials by applying the Rodriguez-Villegas transform to the odd period polynomials for Hecke eigenforms on $\mathrm{SL}_2(\mathbb{Z})$ studied by Conrey et al. [16]. He asked for a generalization for the full period polynomials for such Hecke eigenforms in connection to recent work of El-Guindy and Raji [21]. Theorems 7 and 8 answer this question and provide the generalization for all even weight $k \geq 4$ newforms on congruence subgroups of the form $\Gamma_0(N)$. Theorem 7 additionally offers an explicit combinatorial description of the zeta polynomials in terms of weighted moments. In fact, the developments of [37] are made possible by the recent general proof of the following Riemann Hypothesis for period polynomials.

Theorem 9 ([30]) *For any even integer $k \geq 4$, and any level N, all of the zeros of the period polynomial $R_f(z)$ are on the unit circle.*

Remark 9 The reason that the term "Riemann Hypothesis" is fitting for this result is that the period polynomials satisfy a natural functional equation relating the values at z and $-1/z$ (itself arising from the functional equation for $L(f, s)$), and this result states that all zeros of the period polynomials lie on the line of symmetry for their functional equations.

Remark 10 In fact, the authors of [30] prove a somewhat stronger statement, giving rather precise bounds on the location of the angles of these zeros.

In line with part (5) of Definition 4, we offer a conjectural combinatorial arithmetic-geometric interpretation of the $Z_f(s)$. To this end, we make use of the Bloch–Kato Conjecture, which offers a Galois cohomological interpretation for critical values of motivic L-functions [6]. Here we consider the special case of the critical values $L(f, 1), L(f, 2), \ldots, L(f, k-1)$. These conjectures are concerned with motives $\mathscr{M}_f$ associated with f, but the data needed for this conjecture can be found in the *λ-adic realization* V_λ of $\mathscr{M}_f$ for a prime λ of $\mathbb{Q}(f)$, where $\mathbb{Q}(f)$ is the field generated by the Hecke eigenvalues $a_n(f)$ (where we have $a_1(f) = 1$). The Galois representation V_λ associated with f is due to Deligne, and we recall the essential properties below. For a high-brow construction of V_λ from $\mathscr{M}_f$, we refer to the seminal paper of Scholl [42].

Deligne's theorem says that for a prime λ of $O_{\mathbb{Q}(f)}$ lying above l, there is a continuous linear representation V_λ unramified outside lN

$$\rho_{f,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(V_\lambda),$$

so that for a prime $p \nmid lN$, the arithmetic Frobenius Frob_p satisfies

$$\mathrm{Tr}(\rho_f(\mathrm{Frob}_p^{-1})) = a_p(f), \text{ and } \det(\rho_f(\mathrm{Frob}_p^{-1})) = p^{k-1}.$$

We may also consider the jth Tate twist $V_\lambda(j)$, which is V_λ but with the action of Frobenius multiplied by p^j. After choosing a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable lattice T_λ in V_λ, we may consider the short exact sequence

$$0 \longrightarrow T_\lambda(j) \longrightarrow V_\lambda(j) \xrightarrow{\pi} V_\lambda/T_\lambda(j) \longrightarrow 0.$$

Bloch and Kato define local conditions $H^1_{\mathbf{f}}(\mathbb{Q}_p, V_\lambda(j))$ for each prime p. We let $H^1_{\mathbf{f}}(\mathbb{Q}, V_\lambda(j))$ be the corresponding global object, i.e., the elements of $H^1(\mathbb{Q}, V_\lambda(j))$ whose restriction at p lies in $H^1_{\mathbf{f}}(\mathbb{Q}_p, V_\lambda(j))$. Analogously, we may define $H^1_{\mathbf{f}}(\mathbb{Q}, V_\lambda/T_\lambda(j))$, which is the Bloch–Kato λ- Selmer group. The Šafarevič–Tate group is

$$Ш_f(j) = \bigoplus_\lambda \frac{H^1_{\mathbf{f}}(\mathbb{Q}, V_\lambda/T_\lambda(j))}{\pi_* H^1_{\mathbf{f}}(\mathbb{Q}, V_\lambda(j))}.$$

The Bloch–Kato Tamagawa number conjecture then claims the following.

Conjecture 1 (Bloch–Kato) Let $0 \le j \le k-2$, and assume $L(f, j+1) \neq 0$. Then we have

$$\frac{L(f, j+1)}{(2\pi i)^{j+1}\Omega^{(-1)^{j+1}}} = u_{j+1} \times \frac{\mathrm{Tam}(j+1)\#Ш(j+1)}{\#H^0_{\mathbb{Q}}(j+1)\#H^0_{\mathbb{Q}}(k-1-j)} =: C(j+1).$$

Here, $\Omega^{\pm}$ denotes the Deligne period, Tam denotes the product of the Tamagawa numbers, $H^0_{\mathbb{Q}}$ is the set of global points, and u_{j+1} is a non-specified unit of $\mathbb{Q}(f)$.

Remark 11 Note that $L(f, j+1) \neq 0$ in this range provided that $j+1 \neq k/2$.
We denote the normalized version of $C(j+1)$ by

$$\widetilde{C(j+1)} = C(j+1) \cdot \frac{(i\sqrt{N})^{j+1}\Omega^{(-1)^{j+1}}}{(k-2-j)!}, \tag{36}$$

but when $L(f, j+1) = 0$, we define $\widetilde{C(j+1)} := 0$.

Theorem 10 (Theorem 1.4 of [37]) *Assuming the Bloch–Kato Conjecture and the notation above, we have that*

$$M_f(m) = \sum_{0 \le j \le k-2} \widetilde{C(j+1)} j^m,$$

which in turn implies for each non-negative integer n that

$$Z_f(-n) = \sum_{j=0}^{k-2} \left(\sum_{h=0}^{k-2} \sum_{m=0}^{k-2-h} n^h \binom{m+h}{h} \cdot s(k-2, m+h) \right) j^m \widetilde{C(j+1)}.$$

Finally, we offer an intriguing connection of the zeta polynomials $Z_f(s)$ to the world of combinatorial geometry. This investigation is motivated by the following question. In light of the vast importance of the distribution of the zeros of the Riemann zeta function on (presumably) the critical line, it is natural to consider the distribution of the zeros of $Z_f(s)$ on the line $\mathrm{Re}(s) = 1/2$. Of course, the $Z_f(s)$ are polynomials, and so any interesting distributional properties only make sense

in the context of limits of sequences of modular forms as the level or weights go to infinity. In this sense, one can ask: do their zeros behave in a manner which is analogous to the zeros of the Riemann zeta function $\zeta(s)$? Namely, how are their zeros distributed in comparison with the growth of

$$N(t) := \#\{\rho = s + it \ : \ \zeta(\rho) = 0 \ \text{ with } 0 < t \leq T\},$$

which is well known to satisfy

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)? \tag{37}$$

As we shall see, the zeros of $Z_f(s)$ behave in a manner that is somewhat analogous to (37) in terms of its highest zero.

To make this precise, it turns out that the correct answer lies in comparing $Z_f(s)$ with two families of combinatorial polynomials. In what follows, we note that for $x, y \in \mathbb{C}$, the binomial coefficient $\binom{x}{y}$ is defined by

$$\binom{x}{y} := \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}.$$

We see below that the $Z_f(s)$, depending on $\varepsilon(f)$, can naturally be estimated by the polynomials

$$H_k^+(s) := \binom{s+k-2}{k-2} + \binom{s}{k-2}, \tag{38}$$

$$H_k^-(s) := \sum_{j=0}^{k-3} \binom{s-j+k-3}{k-3}. \tag{39}$$

Theorem 11 (Theorem 1.3 of [37]) *Assuming the notations and hypotheses above, the following are true:*

1. *The zeros of $H_k^-(-s)$ lie on the line $\mathrm{Re}(s) = 1/2$, and they are the complex numbers $\rho = \frac{1}{2} + it$ with $t \in \mathbb{R}$ such that the value of the monotonically decreasing function*

$$h_k(t) := \sum_{j=0}^{k-3} \cot^{-1}\left(\frac{2t}{2j+1}\right)$$

lies in the set $\{\pi, 2\pi, \ldots, (k-3)\pi\}$. Similarly, the zeros of $H_k^+(-s)$ lie on the line $\mathrm{Re}(s) = 1/2$ and have imaginary parts t which may be found by solving for $h_k(t)$ to lie in the set $\{\pi/2, 3\pi/2, \ldots, (k-5/2)\pi\}$. Moreover, as $k \to \infty$, the

highest pair of complex conjugate roots of $H_k^-(s)$ *have imaginary part equal in absolute value to*

$$\frac{(k-3)(k-1)}{2\pi} + O(1),$$

and the height of the highest roots of $H_k^+(s)$ *is*

$$\frac{(k-3)(k-1)}{\pi} + O(1).$$

2. *Let* $f \in S_4(\Gamma_0(N))$ *be a newform. If* $\varepsilon(f) = -1$*, then the only root of* $Z_f(s)$ *is at* $s = 1/2$*. If* $\varepsilon(f) = 1$*, then there are two roots of* $Z_f(s)$*, and as* $N \to \infty$*, their roots converge on the sixth order roots of unity* $\exp(\pm \pi i/3)$.
3. *For fixed* $k \geq 6$*, as* $N \to +\infty$*, the zeros of* $Z_f(s)$ *for newforms* $f \in S_k(\Gamma_0(N))$ *with* $\varepsilon(f) = \pm 1$ *converge to the zeros of* $H_k^{\pm}(-s)$*. Moreover, for all* k, N*, if* $\varepsilon(f) = 1$ (resp. $\varepsilon(f) = -1$)*, then the imaginary part of the largest root is strictly bounded by* $(k-3)\left(k-\frac{7}{2}\right)$ (*resp.* $(k-4)\left(k-\frac{9}{2}\right)$).

Remark 12 Theorem 11 (3) is somewhat analogous to (37). Since the zeros of $Z_f(s)$ are approximated by those of $H_k^{\pm}(-s)$, the analog of $N(T)$ is dictated by Theorem 11 (1), where the largest zero has imaginary part $\sim \frac{k^2}{2\pi}$ or $\sim \frac{k^2}{\pi}$ depending on the sign of the functional equation. Numerical investigations indicate that the locations of the "high" zeros are, however, somewhat differently arranged than those of the Riemann zeta function, the nth highest pair of zeros being at approximately height N_0/n, where N_0 is the highest zero.

As we shall describe shortly, the zeta polynomials $Z_f(s)$ may be thought of, via Theorem 11, as arithmetic-geometric Ehrhart polynomials. In this comparison, the combinatorial structure in Theorem 10, which we call the "Bloch–Kato complex," serves as an analogue of a polytope. Assuming the Bloch–Kato Conjecture, Theorem 10 describes the values $Z_f(-n)$ as combinatorial sums of m-weighted moments of the jth Bloch–Kato components. To describe this combinatorial structure, we made use of the Stirling numbers $s(n,k)$ which can be arranged in a "Pascal-type" triangle

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 1 & & & & \\
 & & & 0 & & -1 & & 1 & & & \\
 & & 0 & & 2 & & -3 & & 1 & & \\
 & 0 & & -6 & & 11 & & -6 & & 1 & \\
0 & & 24 & & -50 & & 35 & & -10 & & 1 \\
0 & -120 & & 274 & & -225 & & 85 & & -15 & 1
\end{array}$$

thanks to the recurrence relation

$$s(n,k) = s(n-1,k-1) - (n-1) \cdot s(n-1,k).$$

This follows from the obvious relation

$$(x)_n = (x)_{n-1}(x-n+1) = x(x)_{n-1} - (n-1)(x)_{n-1}.$$

The Bloch–Kato complex is then obtained by cobbling together weighted layers of these Pascal-type triangles using the binomial coefficients appearing in (34).

The connection to Ehrhart polynomials arises from the central role played by the $H_k^{\pm}(-s)$ in our study of the $Z_f(s)$. In [40], Rodriguez-Villegas proved that certain Hilbert polynomials, such as the $H_k^{\pm}(-s)$, which are Rodriguez-Villegas transforms of $x^{k-2} \pm 1$, are examples of zeta polynomials. These well-studied combinatorial polynomials encode important geometric structure such as the distribution of integral points in polytopes.

Given a d-dimensional integral lattice polytope in $\mathbb{R}^n$, we recall that the Ehrhart polynomial $\mathscr{L}_p(x)$ is determined by

$$\mathscr{L}_p(m) = \#\{p \in \mathbb{Z}^n : p \in m\mathscr{P}\}.$$

The polynomials $H_k^{-}(s)$ whose behavior determines an estimate for those of $Z_f(s)$ (when $\varepsilon(f) = -1$) as per Theorem 11 are the Ehrhart polynomials of the simplex (cf. [5])

$$\operatorname{conv}\left\{e_1, e_2, \ldots, e_{k-3}, -\sum_{j=1}^{k-3} e_j\right\}.$$

We note that in Sect. 1.10 of [26], Gunnells and Rodriguez-Villegas also gave an enticing interpretation of the modular-type behavior of Ehrhart polynomials. Namely, they noted that the polytopes P with vertices in a lattice L, when acted upon by $\mathrm{GL}(L)$ in the usual way, have a fixed Ehrhart polynomial for each equivalence class of polytopes. Hence, these classes may be thought of as points on a "modular curve," and the operation of taking the ℓth coefficient of the Ehrhart polynomial is analogous to a modular form. This analogy is strengthened as they define a natural Hecke operator on the set of Ehrhart polynomials, such that the ℓth coefficients of them are eigenfunctions. Moreover, they show that these eigenclasses are all related to explicit, simple Galois representations. Thus it is natural—as well as intriguing—to speculate about the relationship between these observations and Theorem 11. In particular, we have shown that as the level N of cusp forms of a fixed weight k tends to infinity, the coefficients of the zeta polynomial $Z_f(s)$ tend to (a multiple of) these coefficients of Ehrhart polynomials considered in [26]. It is also interesting to note that Zagier defined (see [49]) a natural Hecke operator on the period polynomials of cusp forms, which commutes with the usual Hecke operators acting on cusp forms. Thus one may ask if there is a reasonable interpretation of Hecke operators on the zeta functions $Z_f(s)$, which ties together this circle of ideas.

In the following sections, we will give a brief outline of the main underlying ideas of the proofs of these results. In particular, we devote the next subsection to some tools needed for the proofs, follow up with the proof sketches, and finish with several examples.

2.3 Tools Needed for the Proof of Manin's Conjecture

2.3.1 Work of Rodriguez-Villegas

We begin our summary of key tools for the proofs of the preceding section by recalling the clever and useful observations of Rodriguez-Villegas in [40]. We will only be concerned with a specialized result; although the reader is referred to the original paper for a more general version and interesting mathematical context of his work. We first take a polynomial $U(z)$ of degree e with $U(1) \neq 0$. The nonvanishing condition is non-essential, as otherwise one simply factors out all the powers of $1-z$ from the polynomial and then applies the Rodriguez-Villegas transform; however, it is convenient for the simplest description to have the exact degree of the polynomial with these factors pinned down. We then define the rational function

$$P(z) := \frac{U(z)}{(1-z)^{e+1}}.$$

Expanding in z, we have

$$P(z) = \sum_{n=0}^{\infty} h_n z^n,$$

and it is easy to see that there is a polynomial $H(z)$ of degree $d-1$ such that $H(n) = h_n$ for all n. The key observation of [40] is then as follows.

Theorem 12 (Rodriguez-Villegas) *If all roots of U lie on the unit circle, then all roots of $Z(s)$ lie on the vertical line $\mathrm{Re}(z) = 1/2$. Moreover, if U has real coefficients and $U(1) \neq 0$, then $Z(s)$ satisfies the functional equation*

$$Z(1-s) = (-1)^{d-1} Z(s).$$

Proof The first claim is simply the special case of the Theorem of [40] when $d = e + 1$. The second claim was described in Sect. 4 of [40], but for the reader's convenience we sketch the proof. By the single proposition of [40], it suffices to show that $P(1/z) = (-1)^d x^{d-e} P(z)$. Now suppose that U factors as

$$U(z) = (z - \rho_1) \ldots (z - \rho_e),$$

where each ρ_j is on the unit circle but not equal to 1. Then

$$z^e U\left(\frac{1}{z}\right) = (1 - z\rho_1) \ldots (1 - z\rho_e).$$

Since the coefficients of U are real, we have $(-1)^e \rho_1 \rho_2 \dots \rho_e = 1$, and dividing by this quantity yields

$$U\left(\frac{1}{z}\right) = z^{-e} U(z).$$

The claimed transformation for P then follows directly from the definition.

2.3.2 Period Polynomials at 1

Here, we recall a simple result that we shall need, related to the order of $R_f(s)$ at $s = 1$.

Lemma 1 *Assuming the notation above, $R_f(1) \neq 0$ if $\varepsilon(f) = 1$, and $R_f(s)$ has a simple zero at $s = 1$ if $\varepsilon(f) = -1$.*

Proof The functional equation for $\Lambda(f, s)$ shows that

$$\begin{aligned} R_f(1) &= \sum_{j=0}^{k-2} \binom{k-2}{j} \Lambda(f, j+1) \\ &= \begin{cases} \Lambda\left(f, \frac{k}{2}\right) + 2\sum_{j=\frac{k+2}{2}}^{k-2} \binom{k-2}{j} \Lambda(f, j+1) & \text{if } \varepsilon(f) = 1, \\ \Lambda\left(f, \frac{k}{2}\right) & \text{if } \varepsilon(f) = -1. \end{cases} \end{aligned} \tag{40}$$

Now $\Lambda(f, s)$ is real-valued on the real line, and well-known work of Waldspurger [46] implies that $\Lambda\left(f, \frac{k}{2}\right) \geq 0$. Furthermore, Lemma 2.1 of [30] states that

$$0 \leq \Lambda\left(f, \frac{k}{2}\right) \leq \Lambda\left(f, \frac{k}{2} + 1\right) \leq \cdots \leq \Lambda(f, k-1), \tag{41}$$

and that $\Lambda\left(f, \frac{k}{2}\right) = 0$ if $\varepsilon(f) = -1$. So, if $\varepsilon(f) = 1$, then the expression in the first case of (40) is composed of all non-negative terms, which cannot all vanish as it is impossible for all periods of f to be zero. Hence, in this case, $R_f(1) \neq 0$. If $\varepsilon(f) = -1$, then as $\Lambda\left(f, \frac{k}{2}\right) = 0$, we see that $R_f(1) = 0$. To see that this zero is simple, note in a similar manner that all terms in $R_f'(1)$ are non-negative, with the last term being $(k-2)\Lambda(f, k-1)$. But this term cannot be zero, as the chain of inequalities in (41) would then imply that all periods of f are zero.

2.3.3 Ingredients for Theorem 10

We first describe the local conditions $H^1_{\mathbf{f}}(\mathbb{Q}_p, V_\lambda(j))$ for a given prime p, following [6, Sect. 3]. Recall that λ was the prime above l in Deligne's representation V_λ.

The first case is when $p = l$. Here, we define

$$H^1_{\mathbf{f}}(\mathbb{Q}_p, V_\lambda(j)) := \ker\left(H^1_{\mathbf{f}}(D_p, V_\lambda(j)) \to H^1_{\mathbf{f}}(D_p, V_\lambda(j) \otimes \mathbb{B}_{\mathrm{cris}})\right),$$

where D_p denotes a decomposition group for a prime over p. For a definition of the $\mathbb{Q}_p$-algebra $\mathbb{B}_{\mathrm{cris}}$, we refer to Berger's article [3, II.3].

For the other cases (i.e., $p \neq l$), we let

$$H^1_{\mathbf{f}}(\mathbb{Q}_p, V_\lambda(j)) := \ker\left(H^1_{\mathbf{f}}(D_p, V_\lambda(j)) \to H^1_{\mathbf{f}}(I_p, V_\lambda(j))\right),$$

where I_p is the inertia subgroup. We let $H^1_{\mathbf{f}}(\mathbb{Q}, V_\lambda(j))$ be the corresponding global object, i.e., the elements of $H^1(\mathbb{Q}, V_\lambda(j))$ whose restriction at p lies in $H^1_{\mathbf{f}}(\mathbb{Q}_p, V_\lambda(j))$.

We note that Bloch and Kato's Tamagawa number Conjecture 1 is independent of any choices, cf. [18, Sect. 6], or for more detail cf. [6, Proposition 5.14 (iii)] and [6, p. 376], in which the independence of the choice in lattice in the Betti cohomology is discussed.

Secondly, we describe the set of global points $H^0_{\mathbb{Q}}$, with the appropriate Tate twists:

$$H^0_{\mathbb{Q}}(j) := \bigoplus_{\lambda} H^0(\mathbb{Q}, V_\lambda / T_\lambda(j))$$

2.4 *Sketch of Proofs for Theorems on Zeta Polynomials and the Riemann Hypothesis for Period Polynomials*

Here we sketch the proofs of the theorems in Sect. 2.2. We begin with the proof of the Riemann Hypothesis for period polynomials, which is an essential ingredient in the main proofs about zeta polynomials.

Proof (Sketch of the Proof of Theorem 9) The general strategy is to consider, for $m := \frac{k-2}{2}$, the polynomial

$$P_f(X) := \frac{1}{2}\binom{2m}{m} \Lambda\left(f, \frac{k}{2}\right) + \sum_{j=1}^{m} \binom{2m}{m+j} \Lambda\left(f, \frac{k}{2} + j\right) X^j.$$

The result then follows if the unit circle contains all of the zeros of

$$T_f(X) := P_f(X) + \varepsilon(f) P_f(1/X).$$

Following a standard trick, we substitute $X \to z = e^{i\theta}$, note that $T_f(i\theta)$ takes on only real values, and hope to find enough sign changes to locate all zeros using the Intermediate Value Theorem. In fact, it turns out that $T_f(z)$ is a trigonometric polynomial in sin or cos depending on $\varepsilon(f)$. For large values of k or N, results of

Pólya [39] and Szegö [44] then give strong conditions on the locations of the zeros, given unimodal properties of the coefficients of these polynomials. Overall, this boils the argument down (in the limit) to proving several handy inequalities for the competed critical L values of f. These are shown to follow from several arguments making keen use of the famous Hadamard factorization for $\Lambda(f,s)$ in terms of the roots of Λ.

This proof is only sufficient for modular forms of large weight or level. For the remaining cases, separate arguments can be directly constructed (for example, as we will look at in Sect. 2.5, if the weight is a small number like 4, the polynomial is only of degree 2 and can hence be studied somewhat directly). After this, finitely many cases remain, which may then be checked numerically, as the authors of [30] do using SAGE.

We proceed to outline the proof of Theorem 7, which gives the version of the Riemann Hypothesis inherited by $Z_f(s)$ from the Riemann Hypothesis for $P_f(s)$. Firstly, however, we need the result of Theorem 8.

Proof (Proof of Theorem 8) This proof is a combinatorial exercise; we refer the reader interested in the details to [37].

Proof (Proof of Theorem 7) We first let

$$\widehat{R}_f(z) := \frac{R_f(z)}{(1-z)^{\delta_{-1,\varepsilon(f)}}},$$

where $\delta_{i,j}$ is the Kronecker delta function. By Theorem 9 and Lemma 1, we see that $\widehat{R}_f$ is a polynomial of degree $k-2-\delta_{-1,\varepsilon(f)}$, all of whose roots lie on the unit circle, and such that $\widehat{R}_f(1) \neq 0$. Thus we have

$$\frac{R_f(z)}{(1-z)^{k-1}} = \frac{\widehat{R}_f(z)}{(1-z)^{k-1-\delta_{-1,\varepsilon(f)}}}.$$

Applying Theorems 8, 9, and 12 with $d = k-1-\delta_{-1,\varepsilon(f)}$ yields the result, and in particular shows that the zeros of $Z_f(s)$ lie on the line $\mathrm{Re}(s) = 1/2$.

Proof (Proof of Theorem 10) The proof of Theorem 10 follows immediately from replacing the terms involving $L(f,j+1)$ by $\widetilde{C(j+1)}$, and appropriate normalizations.

Proof (Sketch of the Proof of Theorem 11) Part (1) follows directly from the observation (which is easy to combinatorially verify) that the polynomials $H_k^-(x)$ are Rodriguez-Villegas transforms of $\sum_{j=0}^{k-3} x^j$ and that the $H_k^+(x)$ are the transforms of $x^{k-2}+1$.

The fact that the zeros of $H_k^-(x)$ lie on the line $\mathrm{Re}(x) = 1/2$ is one of the main examples of Theorem 12 in [5]. The precise location of its zeros in Theorem 11 is a direct restatement (and slight modification in the case of $\varepsilon(f) = +1$) of Theorem 1.7 of [5].

The proof of part (2) follows directly from the more precise estimates on the locations (and in particular the equidistribution) of the zeros in Theorem 9, which is hinted at in Remark 10 and provided in Theorem 1.2 in [30]. As extracting roots of a polynomial is continuous in the coefficients of the polynomial, we have the desired convergence of the roots of $Z_f(s)$ in the limit to those of the polynomials $H_k^{\pm}(x)$.

Part (3) also follows by utilizing the precise estimates of Theorem 1.2 of [30] to determine the zeros of $R_f(z)$ to high accuracy. The strict upper bound on the imaginary parts of roots follows from a careful consideration of Theorem 1 of [10], using the positivity properties of critical completed L-values reviewed in Lemma 1. When $\varepsilon(f) = -1$, by Lemma 1 and the functional equation for $\Lambda(f, s)$, we see that $R_f(x)/(1-x)$ is a polynomial with all non-positive coefficients, so the Rodriguez-Villegas transform of R_f is the same as the Rodriguez-Villegas transform of this polynomial with the degree lowered by 1. This shows that we may again use Theorem 1 of [10], by applying it to the polynomial $R_f(s)/(x-1)$ with positive coefficients and whose transform has the same zeros as $Z_f(s)$.

2.5 *Examples*

2.5.1 Period Polynomials in Weight 4

To give a flavor of the types of analysis going into the proof of Theorem 9 for small weights and levels, here we consider the case of weight 4 newforms. The period polynomial R_f is then quadratic, and we are concerned with the zeros of

$$-2L(f,1)\pi^2 X^2 + 2\pi i L(f,2)X + L(f,3) = 0.$$

Clearly, we may then apply the quadratic formula. Now, the values $L(f,1)$ and $L(f,3)$ are also related via the functional equation for $L(f,s)$, and the conclusion is thus trivial if $L(f,2) = 0$.

In the case when $L(f,2) \neq 0$, the theorem is equivalent to the estimate

$$\frac{N}{\pi^2} L(f,3)^2 \geq L(f,2)^2.$$

This can be shown using the Hadamard factorization of $\Lambda(f,s)$:

$$\Lambda(f,s) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \exp(s/\rho)$$

Here the product is over all the zeros of $\Lambda(f,s)$ (that is, the non-trivial zeros of $L(f,s)$), and A and B are constants. Now we always have $3/2 \leq \mathrm{Re}(\rho) \leq 5/2$, which means that $\Lambda(f,3) \geq \Lambda(f,2)$, and is enough to imply the desired inequalities.

2.5.2 Zeta Function for the Modular Discriminant

It is interesting to consider the first level 1 newform, namely the normalized Hecke eigenform $f = \Delta \in S_{12}(\Gamma_0(1))$. In this case, $\varepsilon(f) = 1$ and we have

$$R_\Delta(z) \approx 0.114379 \cdot \left(\frac{36}{691} z^{10} + z^8 + 3z^6 + 3z^4 + z^2 + \frac{36}{691} \right)$$
$$+0.00926927 \cdot (4z^9 + 25z^7 + 42z^5 + 25z^3 + 4z).$$

The ten zeros of R_Δ lie on the unit circle, and are approximated by the set

$$\{\pm i, -0.465 \pm 0.885i,\ -0.744 \pm 0.668i,\ -0.911 \pm 0.411i,\ -0.990 \pm 0.140i\}.$$

These are illustrated in the following diagram (Fig. 1).

It is quite tempting from this picture to conjecture that the zeros of R_f always lie on one half of a unit circle. However, this small example is somewhat misleading: the zeros for large level or weight modular forms eventually have to equidistribute on the whole unit circle, as shown in [30]. Another manner in which this example is atypical is the following. The size of the coefficients of R_f follows a unimodal pattern. The same equidistribution-of-zeros result shows that eventually the coefficients of R_f must be unimodal, but in the opposite direction! In fact, if the direction of this unimodality of coefficients were reversed, the Riemann Hypothesis for R_f would follow directly from its functional equation (showing it is a self-inversive polynomial) along with elementary results such as Theorem 1 of [14]. It is this

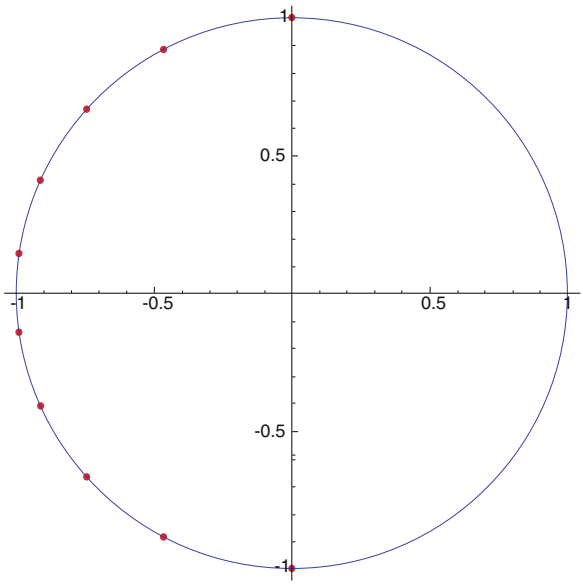
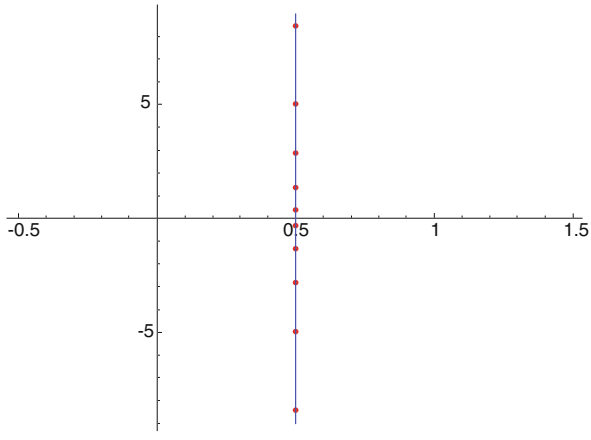


Fig. 1 The roots of $R_\Delta(z)$

Fig. 2 The roots of $Z_\Delta(s)$



eventual "correct" unimodality that follows from the results of [30], and this drastic switching of phenomena that make the proof of Theorem 9 split into several cases and make its statement surprising. In fact, this line of reasoning allows one to conclude that for large weights and levels, the location of the zeros of $R_f(s)$ on the unit circle is a relatively "stable" phenomenon, while a quick numerical experiment for a polynomial such as R_Δ shows that the location of its zeros on the circle is very unstable, and hence rather astounding.

We now consider the Rodriguez-Villegas transform; letting $s \mapsto -s$ we find that

$$\begin{aligned} Z_\Delta(s) \approx\ & (5.11 \times 10^{-7})s^{10} - (2.554 \times 10^{-6})s^9 + (6.01 \times 10^{-5})s^8 - (2.25 \times 10^{-4})s^7 \\ & +0.00180s^6 - 0.00463s^5 + 0.0155s^4 - 0.0235s^3 \\ & +0.0310s^2 - 0.0199s + 0.00596. \end{aligned}$$

Theorem 7 establishes that the zeros ρ of Z_Δ satisfy $\mathrm{Re}(\rho) = 1/2$; indeed, they are approximately

$$\left\{ \frac{1}{2} \pm 8.447i, \frac{1}{2} \pm 5.002i, \frac{1}{2} \pm 2.846i, \frac{1}{2} \pm 1.352i, \frac{1}{2} \pm 0.349i, \right\},$$

as illustrated in the next figure (Fig. 2).

2.5.3 Ehrhart Polynomials and Newforms of Weight 6

Here we consider newforms $f \in S_6(\Gamma_0(N))$ with $\varepsilon(f) = -1$. By Theorem 11 (2), the roots of $Z_f(s)$ may be modeled (with improving accuracy as $N \to \infty$) on the roots of the Ehrhart polynomial of the convex hull

$$\mathrm{conv}\,\{e_1, e_2, e_3, -e_1 - e_2 - e_3\}.$$

The following image displays this tetrahedron.

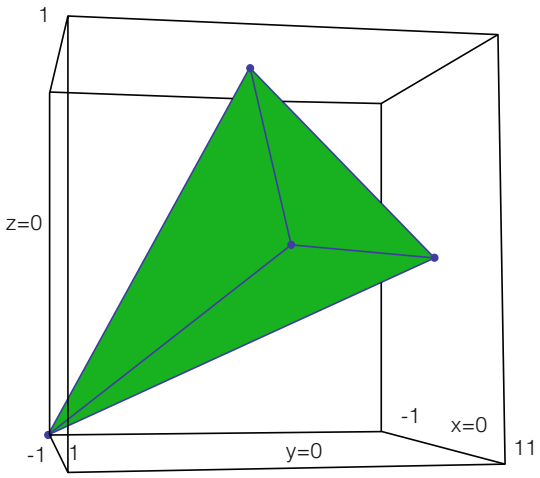


Fig. 3 The tetrahedron whose Ehrhart polynomial is $H_6^-(s)$

The corresponding Ehrhart polynomial counts the number of integer points in dilations of Fig. 3, and is given by the Rodriguez-Villegas transform of $1+x+x^2+x^3$. Namely, we have

$$H_6^-(s) = \binom{s+3}{3} + \binom{s+2}{3} + \binom{s+1}{3} + \binom{s}{3} = \frac{2}{3}s^3 + s^2 + \frac{7}{3}s + 1.$$

Therefore, we find that

$$\lim_{N\to+\infty} \widetilde{Z}_f(s) = \widetilde{H_6^-}(-s) = \left(s - \frac{1}{2}\right)\left(s - \frac{1}{2} + \frac{\sqrt{-11}}{2}\right)\left(s - \frac{1}{2} - \frac{\sqrt{-11}}{2}\right),$$

where the limit is over newforms $f \in S_6(\Gamma_0(N))$ with $\varepsilon(f) = -1$, and where the polynomials with tildes have been normalized to have leading coefficient 1.

2.6 *Concluding Remark*

In both this and the previous section, we have surveyed a number of natural partition-theoretic forms emerging from the same stellar nursery as the Riemann zeta function. Indeed, it is evident that classical zeta functions represent a relatively small subclass in the vast universe of partition-theoretic objects.

Acknowledgements The authors thank Armin Straub for useful discussion on the history of Theorem 5. Ken Ono is supported by National Science Foundation and the Asa Griggs Candler Fund. Larry Rolen thanks the support of the DFG and the University of Cologne postdoc program.

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Reading Riemann

S.J. Patterson

Abstract This is an essay about the reputation that Riemann has acquired as a difficult writer, a reputation which has, if anything, grown over the years. An awareness of the developments in mathematics, and, in particular, in mathematical physics at that time shows that Riemann was working in a very specific environment and that his thinking reflects this environment. It is therefore helpful to know something of this background. Riemann's short paper on the distribution of prime numbers has been the subject of considerable discussion and we shall consider this paper in particular.

1 Introduction

Riemann has been considered a "difficult" author with some commentators on the verge of considering him to be a mathematical Heraclitus whose writings were described as "dark" in classical times (at least by Cicero). One particularly flagrant example of this is Lars Ahlfors' discussion of Riemann's dissertation in a volume published to mark its centenary.[1] Ahlfors writes:

> The most astonishing feature in Riemann's paper is the breathtaking generality with which he attacks the problem of conformal mapping. He has no thought of illustrating his methods by simple examples which to lesser mathematicians would have seemed such an excellent preparation and undoubtedly would have helped his paper to much earlier recognition. On the contrary, Riemann's writings are full of almost cryptic messages to the future. For instance, Riemann's mapping theorem is ultimately formulated in terms which would defy any attempt of proof, even with modern methods.

[1] L. Ahlfors, Developments of the theory of conformal mapping and Riemann surfaces through a century, In: *Contributions to the theory of Riemann surfaces*, Eds. L. Ahlfors, E. Calabi, M. Morse, L. Sario, D. Spencer, Annals of Mathematics Studies, 30, Princeton UP, 1953.

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H. Montgomery et al. (eds.), *Exploring the Riemann Zeta Function*,
DOI 10.1007/978-3-319-59969-4_11

Another curious example is a statement by Hans Freudenthal[2]

> Riemann's style, influenced by philosophical reading, exhibits the worst aspects of German syntax; it must be a mystery to anyone who has not mastered German.

This has to be contrasted with a statement in the preceding paragraph of the same article:

> Riemann's evolution was slow and his life short. What his work lacks in quantity is more than compensated for by its superb quality. One of the most profound and imaginative mathematicians of all time, he had a strong inclination to philosophy, indeed, was a great philosopher, Had he lived and worked longer, philosophers would acknowledge him as one of them. His style was conceptual rather than algorithmic and to a higher degree than that of any mathematician before him. He never tried to conceal his thought in a thicket of formulas. After more than a century his papers are still so modern that any mathematician can read them without historical comment, and with intense pleasure.

To the present author's taste Riemann's writing is clear. However, it cannot be denied that there are difficulties, but this is by no means uncommon in reading authors of a different epoch. Many authors, and Riemann is included here, do not give references for all the background material that is used in a paper. Indeed the ideas may well be "folk-lore" and this is by no means limited to historical authors—as one well knows. Riemann was a mathematician, or, perhaps more precisely, a mathematical physicist, who was mathematically ambitious and employed the most modern tools at his disposal. The present-day reader is confronted with the task of reconstructing the knowledge and thought-patterns of a sophisticated mathematician of the 1850s and 1860s. It is not possible to understand Riemann in isolation—despite what Freudenthal writes at the end of the second passage quoted; both the group of older mathematicians in Göttingen and Berlin who influenced him and the published literature of the time are the context of Riemann's work.

Ahlfors indicates that in his dissertation Riemann did not give examples; what he leaves to one side is whether Riemann *thought* in terms of examples. In the case of the dissertation he took over the work of his predecessors, particularly on the geometrical side Gauss' "Allgemeine Auflösung der Aufgabe die Theile einer gegebenen Fläche etc." (1822/1825)[3] and his geometrical conceptions started from here.[4] He was also clearly familiar with Cauchy's work, much of which was decades old by this time.[5] In other matters, for example, in the theory of non-linear hyperbolic differential equations and the theory of shock waves, Riemann

[2]Riemann, Georg Friedrich Bernhard, *Complete Dictionary of Scientific Biography*, 2008 Charles Scribner's Sons.

[3]Gauss, *Werke* 4, 189–216.

[4]See Henri Paul de Saint-Gervais, *Uniformization of Riemann Surfaces, Revisiting a hundred-year-old theorem,* European Mathematical Society, 2016, esp. Ch. 1.

[5]For the chronology of Cauchy's work see F. Smithies, *Cauchy and the creation of complex function theory,* Cambridge U.P., 1997.

had no compunction about working from an example. Indeed he writes in the "Selbstanzeige der vorstehenden Abhandlung"[6]

> Wie für die Integration der linearen partiellen Differentialgleichungen nicht durch die Entwicklung des allgemeinen Begriffs dieser Aufgabe gefunden worden, sondern vielmehr aus der Behandlung specieller physikalischer Probleme hervorgegengen sind, so scheint auch für die Theorie der nichtlinearen partiellen Differentialgleichungen durch eine eingehende, alle Nebenbedingungen berücksichtigende, Behandlung specieller physikalischer Probleme am meisten gefördert zu werden, und in der That hat die Lösung der ganz speciellen Aufgabe, welche Gegenstand dieser Abhandlung bildet, neue Methoden und Auffassungen erfordert, und zu Ergebnissen geführt, welche wahrscheinlich auch bei allgemeineren Aufgaben eine Rolle spielen werden.[7]

Generally speaking the views expressed by mathematicians as to their principles or methods frequently diverge from their œuvre as a whole and represent their thoughts simply at the time of writing. We recall in this context that Klein was very impressed by how elegantly Riemann's theory of the *P*-function led to Kummer's results on the hypergeometric function and stressed the geometrical and conceptual aspects of his function theory. C.L. Siegel, on the other hand, in his paper on the Riemann Nachlass[8] stresses Riemann's formal virtuosity, an aspect which has been reinforced by recent work of W. Gabcke (private communication). No mathematician of Riemann's stature can be categorized[9] by one or two simplistic phrases.

Riemann's dissertation and most of his other published mathematical papers were taken up relatively soon by young mathematicians like H.A. Schwarz, C. Neumann and later, from about 1880 onwards, by F. Klein and H. Poincaré; his work on theta functions and other aspects of the theory of algebraic functions was taken up by J. Thomae, G. Roch, A. Clebsch, P. Gordan, and others. Indeed Riemann's theory ousted K. Weierstrass' more algebraic approach to the theory of abelian functions which meant that the construction of an algebraic theory, which made no use of analytic techniques, was only completed in the 1940s, mainly by A. Weil.[10] Riemann's work on trigonometrical series was taken up by G. Cantor at

[6] [1, p. 208].

[7] [1, p. 208], p. 208—in English "Just as the integration of linear partial differential equations was not found by developing general concepts but rather through the treatment of special physical problems, it seems that also the theory of non-linear partial differential equations will profit most from a deeper treatment of special physical problems involving all aspects of those problems and indeed the solution of the problem which is the subject of this memoir has led to the development of new methods and concepts and to results which will probably play a role in more general investigations".

[8] [1, pp. 770–805, esp. p. 771].

[9] As an historical aside the word "category", or rather its Greek precursor, was first used in its present sense by Aristotle and originally meant "accusation". It is derived from "kata", down, and "agora", the market place/assembly place.

[10] As it happens Weil does not seem to have had any interest in Weierstrass' work. In his thesis, A. Weil, L'arithmétique sur les courbes algébriques, *Acta Math.* 52 (1928) 1–35(= *Œuvres scientifiques,I*,11–45) he uses the theory of theta functions to develop what he needs about the Jacobian variety. See p. 26ff. C.L. Siegel who followed this up a few years later with his theorem on integral points on curves, in über einige Anwendungen diophantischer Approximationen, *Abh.*

the suggestion of Weierstrass and was the subject of many investigations. Likewise Riemann's theory of linear and non-linear waves in air was taken up in the 1870s by Rankine and Rayleigh. His work in general was not seen as incomprehensible, merely as a challenge to the upcoming generation of mathematicians. Indeed W. Purkert[11] has listed over 550 papers influenced by Riemann which appeared in the 25 years after his death, i.e. to the point when Riemann would have turned 65. It should be added that Riemann would not have been appointed as Dirichlet's successor in Göttingen if his work had been a collection of "cryptic messages to the future".

The role of Felix Klein in understanding Riemann is important. He "discovered" Riemann for himself around 1880 and his little book on Riemann's function theory[12] shows in detail how Klein approached Riemann's work. He had from then on the enthusiasm of a convert. When after his collapse of 1882 he went in 1886 to Göttingen it was to build up that university as a "centre of excellence" in mathematics inside the Prussian state. Klein used the myths of both Gauss and Riemann to further this project and had a very long-lasting influence on the perception of both mathematicians.[13]

Riemann's paper on prime numbers seems to be something of an exception in that it took over 30 years before J. Hadamard made the first substantial progress.[14] Riemann's work was by no means forgotten in the intervening years and his ideas were quoted by a number of authors. There seems to have been a particular revival after the publication of Riemann's *Gesammelte mathematische Werke* in 1876 but it took some time before this interest bore fruit.

Before this event there had been a considerable interest in prime numbers both with Chebyshev's work and its further developments, and in the calculation of extensive tables. H.J.S. Smith, who was very familiar with the contemporary continental mathematical literature, wrote in his Presidential Address to the London Mathematical Society in 1876, so around the time that Riemann's *Werke* appeared, the following passage[15]:

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d, Preußischen Akad. d. Wiss., Phys.-math. Klasse, 1929, Nr. 1 =(*Ges. Abh. I*, 209–266) used the same methods—see p. 44ff. When Weil came to develop the theory of Jacobian varieties he seems to have taken his thesis as his starting point. Siegel in later years was much more interested in Weierstrass' work but was never interested in the general theory of Jacobian varieties.

[11] [1, p. 870ff.].

[12] F. Klein, *On Riemann's theory of algebraic functions and their integrals,* Dover, 1963—German edition, 1881, English translation, 1893.

[13] Apart from his booklet on Riemann's function theory Klein also wrote a very interesting essay on Riemann, delivered as an address in to a meeting of the Naturforscherversammlung zu Wien on Riemann's birthday, 1894—see his *Gesammelte Mathematische Abhandlungen, Bd. 3,* pp. 482–497.

[14] For the history we refer the reader to Landau's *Handbuch*.

[15] This address was read on the 9th November 1876, so late in the year. He may therefore already have been in possession of a copy of the *Werke*. On the other hand, we know that Smith was very well informed about developments in number theory outside the UK as his Reports to the British Association testify.

> As to our knowledge of the series of the prime numbers themselves, the advance since the time of Euler has been great, if we think of the difficulty of the problem; but very small if we compare what has been done with what still remains to do. We may mention, in the first place, the undemonstrated, and indeed conjectural, theorems of Gauss and Legendre as to the asymptotic value of the number of primes inferior to a given limit x (the former theorem assigns for this value the integral logarithm $\mathrm{Li}(x) = \int_0^x \frac{dx}{\log x}$, the latter an expression of the form $\frac{x}{\log x - a}$). But these theorems are only approximately consistent one another, and are perhaps still less approximately true. The memoir of Bernhard Riemann, "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse", contains (so far as I am aware) the only investigation of the asymptotic frequency of the primes which can be regarded as rigorous.

It is clear that the word "rigorous" here has a rather different meaning to that used today. It is fairly clear that Smith does not mean "indisputable" but rather "theoretical" (or even 'heuristic') in contradistinction to "empirical". These notions were not used then in the strict sense we employ today.

If one examines the development of complex analysis in the second half of the nineteenth century, then one sees that it took quite some time before many of the ideas inherent in Riemann's papers, for example, a flexible use of contour integrals, of which the Pochhammer integrals, due to C. Jordan (1887) and L. Pochhammer (1890) or the use of the principle of the argument[16] only slowly became common knowledge and practice. These developments had to be achieved before the paper on prime numbers could be properly digested. We shall see that it was especially Riemann's formula for $N(T)$ that caused difficulty at the time, much more so than the actual calculation of the zeros.

2 Prime Number Theory Before Riemann's Paper

The first published discussion of the distribution of the primes, apart from Euclid's, was, as is well-known, in the first edition (1798) of Legendre's *Essai sur la théorie des nombres* in which he suggested the well-known asymptotic law $\pi(X) \sim \frac{X}{\log X}$ where, as usual, $\pi(X)$ denotes the number of primes $\leq X$. This approximation is numerically not entirely convincing and Legendre attempted to find a better approximation but his attempt went in the wrong direction. The attempt reflects the fact that the approximation $\frac{X}{\log X}$ is numerically not all that good; indeed the approximation by the logarithmic integral is also not particularly good. After Legendre's work there was, for a long time, little published on these questions. The next major advance in the theory of primes was Dirichlet's theorem on primes in

[16]The principle of the argument seems to have been very little used in the second half of the nineteenth century despite the fact that it is a part of Riemann's approach to the theory of theta functions. For some details of the history see F. Smithies, *loc. cit.* p173ff. It seems to have attracted little attention for quite some time; apparently a topological approach was unfamiliar to most mathematicians of the early and middle nineteenth century.

arithmetic progressions. This proof, presented to the Berlin Academy in 1837 and published in 1839, was a remarkable achievement. Dirichlet expected that he would be able to prove Legendre's empirical law by the same methods (see the discussion in Landau's *Handbuch*) but this proved illusory. Ten years later the young Riemann spent 2 years in Berlin and seems to have learnt of Dirichlet's hopes.

In 1848 the first real progress towards a proof of what we may call "the prime number theorem" was made by Chebyshev. His first paper on the subject seems to have been read to the Petersburg Academy on 24th May 1848 and published in Russian in 1849 and in French in 1851; it was reprinted in Liouville's *Journal de mathématiques pures et appliquées* in 1852. In this paper Chebyshev uses, as one may formulate it, the theory of the zeta function as a function of a real variable combined with partial summation to show that if $\frac{\pi(X)}{X/\log X}$ tends to a limit, then this limit is 1. He also discusses Legendre's empirical law and argues that the better approximation should be $\mathrm{Li}(X)$ rather than $\frac{X}{\log X}$. In a second paper which appeared in Liouville's *Journal* in 1852 directly following his first paper, but only 2 years later in the *Mémoires* of the Petersburg Academy he introduces his method of combining the prime decomposition of $n!$ with Stirling's formula to prove Bertrand's Postulate.

More or less simultaneously with Chebyshev's work an English barrister and professor of law at University College, London, Charles J. Hargreave published a paper in the *Philosophical Magazine* in which he applies arguments similar to those of Chebyshev's first paper to the distribution of prime numbers.[17] It seems that this was entirely coincidental. Hargreave's paper is in parts rather verbose and by no means so elegant or profound as that of Chebyshev, which would have difficult for anyone to achieve, but he too came to the conclusion that $\mathrm{Li}(X)$ should be the better approximation to $\pi(X)$ and discusses numerical evidence to support this thesis.

On Christmas Eve, 1849 Gauss wrote a letter to his erstwhile student Johann Franz Encke (1791–1865) who was at this time a well-established professor of astronomy in Berlin and director of the observatory there. In this letter, which has been printed in Vol. 2. of Gauss' *Werke*, Gauss explains how he has begun to investigate the distribution of the primes in 1792 or 1793 and had come to similar conclusions. One may well ask, why he did this at this point in time and why in a letter to Encke, who, as far as we know, had little interest in the theory of numbers.

Gauss had a tendency to sit on his more mathematical results for long periods of time until someone else had either rediscovered them or was close to doing so. He behaved in a very ungenerous fashion towards Legendre as regards the method of least squares and towards János Bolyai, the son of his closest friend from his student days, in the matter of non-euclidean geometry. He behaved in a similar fashion towards Jacobi and Abel when they developed the first theory of elliptic functions and towards Jacobi when he constructed Jacobi sums and began to make progress towards the biquadratic and cubic reciprocity laws. Jacobi, who had a sharp tongue, made his irritation about Gauss' behaviour clear. In a letter to Bessel from 3rd April 1835 he wrote "Bei Gauss heisst es nicht: *de mortuis nil nisi bene*, sondern *de*

[17] Vol. 35 (1849) 36–53. This paper is in the bibliography of Landau's *Handbuch*.

mortuis et de vivis nil", i.e. "Gauss' motto is not "nothing but good about the dead" but rather "nothing about the dead or the living".[18] He also wrote to his brother Moritz Jacobi on the 21st September 1849,"Er [Gauss] hat in den 20 Jahren weder mich noch D[Dirichlet] jemals citiert", i.e. He [Gauss] has not cited either me or Dirichlet in the [last] 20 years".[19] Gauss attempted to rationalize his behaviour from time to time, for example, with his motto *pauca sed matura* but his explanations are unconvincing. Rather Gauss was reluctant to share with others the renown coming from certain advances. In this he was not unlike Newton in his relationship to Hooke and to Leibniz.

We do not know whether Gauss had learnt of Chebyshev's work by the end of 1849. He had had a number of contacts with Petersburg but there seems to be nothing in his papers that would decide the question one way or the other. However, the *Philosophical Magazine*, in which Hargreave's article appeared, was subscribed to in Göttingen and we know that Riemann, at least, read articles in it.[20] We can conclude that Riemann had at least seen this paper and that Gauss had had the opportunity to see it.

Chebyshev visited Berlin in the summer of 1852 and conferred with Dirichlet. Later that year Dirichlet spent some time in Göttingen, as we know from Dedekind's Lebenslauf of Riemann. He spent time together with Riemann and so one can presume that Riemann was also informed about Chebyshev's work at the latest by 1852.[21] It seems therefore reasonable to conclude that the seeds for Riemann's thinking about primes had been sown by then.

3 In the Intermezzo

Between Riemann's paper and Hadamard's work on the zeta function more than 30 years elapsed. During most of this period one can find signs of interest in Riemann's paper. A. Genocchi published a Italian version of Riemann's paper in 1860.[22] This is an interesting reworking of Riemann's explicit formula. Genocchi discusses the experimental evidence for the prime number theorem. It seems as if he would have been dissatisfied with an error term as large as $\mathcal{O}(X^{\frac{1}{2}})$. In this he is reflecting Riemann's own opinion expressed in the last paragraph of his paper where he also

[18] K.-R. Biermann, Die Beziehungen zwischen C.F. Gauss und F.W. Bessel, *Mitt. Gauss-Ges. Göttingen*, 3 (1966) 7–20, esp. p. 18.

[19] *loc. cit.*

[20] See his "Selbstanzeige der vorstehenden Abhandlungen", [1, p. 208]. In fact one of the articles quoted there is in the same volume as Hargreave's paper.

[21] See also Harold Edwards, *Riemann's zeta function*, p. 5.

[22] A. Genocchi, Fermole per determinare quanti siano i numeri primi ad un dato limite, *Ann. di Mat. pura et applicata,* Ser. 1, 3 (1860) 52–59.

tries to explain why $\mathrm{Li}(X) - \pi(X)$ is always positive.[23] These considerations were rendered null and void by Littlewood's theorem of 1914[24] in which he showed that the sign of the difference changes infinitely often. Genocchi does not discuss the location of the zeros.

From 1874 onwards F. Mertens published a substantial series of papers dealing with prime number theory. At this time there were further, and more accurate, tables produced of prime numbers in which the validity of the conjectured prime number theorem was tested. Some felt the deviations were too large and looked for alternatives. At the latest after Riemann's *Werke* appeared the function discussed there, namely $\mathrm{Li}(x) - \frac{1}{2}\mathrm{Li}(x^{\frac{1}{2}}) - \frac{1}{3}\mathrm{Li}(x^{\frac{1}{3}}) - \frac{1}{5}\mathrm{Li}(x^{\frac{1}{5}}) + \frac{1}{6}\mathrm{Li}(x^{\frac{1}{6}}) - \cdots$ was examined to see if it were a better approximation, which, in the ranges considered, it appears to be.[25]

From the middle of the 1880s on some Danish mathematicians began to compute the zeros of the zeta function on the critical line. First, in a rough and unpublished calculation E. Meissel[26] calculated a number of terms in the Taylor expansion of ξ around $s = \frac{1}{2}$ and used this to compute some zeros. This computation was effected by carrying out a numerical integration of using Riemann's integral representation of this function. Meissel also computed $\pi(X)$ for some large values of X.

Around the same time J.L.W.V. Jensen[27] used the expansion

$$(s-1)\zeta(s) - 1 = \sum_{m \geq 1} m^{1-s} \left(\left(1 + \frac{1}{m}\right)^{1-s} - 1 - \frac{1-s}{m} \right)$$

which converges in $\mathrm{Re}(s) > 0$ and can be used to give an "elementary" analytic continuation of the zeta function, to compute the first 9 terms of the expansion around $s = 1$. He derived from this a rough estimate for the first zero.

Later, in 1895, Gram[28] used the formula, valid in $\mathrm{Re}(s) > 0$,

$$\zeta(s) = \lim_{N \to \infty} \sum_{1 \leq m \leq N} m^{-s} - \frac{N^{1-s}}{1-s}$$

[23] At this point Riemann also describes the approximation $\mathrm{Li}(X)$ to $\pi(X)$ as "known".

[24] J. E. Littlewood, Sur la distribution des nombres premiers.*C. R. Acad. Sci. Paris* 158 (1914), 1869–1872.

[25] See W.W. Johnson, Mr James Glaisher's factor tables and the distribution of primes, *Ann. of Math.* 1 (1884) 15–28.

[26] See S.J. Patterson, The Riemann Hypothesis – a short history, In: *Casimir Force, Casimir Operators and the Riemann Hypothesis,* Eds. G. v. Dijk and M. Wakayama, de Gruyter, 2010, pp. 29–41.

[27] J.L.W.V. Jensen, Sur le fonction $\zeta(s)$ de Riemann, *Comptes rendus Acad. Sci. Paris* 104 (1887) 1156–1159. In J.-P. Gram, Note sur la calcul de la fonction $\zeta(s)$, *Oversigt over det Koniglige Danske videnskabernes Selskabs Forhandlinger og dets Medlemmers Arbejter,* 1895, 303–308, the author reports that he had received in 1887 a letter from T.J. Stieltjes who had also undertaken calculations and had found the approximation 14.5 for the first zero.

[28] *loc. cit.*

combined with the Euler-Maclaurin for the same purposes. Later, in 1903, he augmented this with a summation formula due to Abel. The Finnish mathematician Ralf Josef Backlund took up the idea in his thesis of 1916 but using a variant summation formula due to Lindelöf.

It seems as this work was the inspiration for Hardy and Littlewood's "approximate function equation" of 1923. As it turned out Riemann had already found a more precise formula but which he had not published.[29]

What these authors were not in a position to do was to verify that all the zeros of the zeta function, in the regions examined, were on the critical line. It was only after v. Mangoldt gave a proof of the precise version of Riemann's estimate for $N(T)$ in 1905 that it became possible to test the Riemann Hypothesis. He had given a weaker version in 1895 but apparently no-one attempted to develop it numerically. The tests of the Riemann Hypothesis began with the work of Backlund who developed an effective version of v. Mangoldt's formula (1918).

Finally it is of some interest to read what a well-educated young mathematician, G.B. Mathews (born in 1861) wrote about Riemann's paper in 1892, that is, just before Hadamard's contributions began[30]:

> [That the formula $\pi(X) = \mathrm{Li}(X)$] is approximately [i.e. empirically] correct, does not in any way prove that this is a proper asymptotic formula. It seems clear that Gauss was led to it simply by observation, and it does not appear that he ever accounted for it in any theoretical way, The only satisfactory attempt to determine a general analytical formula for $F(x)$ [i.e. $\pi(X)$] appears to be that contained in Riemann's celebrated memoir. This is confessedly incomplete, and the analysis which it contains is very peculiar and difficult: but because of its great importance, some account of it ought to be given. On the properties of the function $\Gamma(z)$ for a complex variable, which will have to be assumed in the course of the investigation, the reader may consult Prym, *Zur Theorie der Gammafunction* (Crelle, lxxxii. 165) and various papers by Bourguet and Mellin in vols. i, ii, iii, and viii of the Acta Mathematica.[31]

One can summarize these authors by saying that Riemann's work was known and appreciated by a fair number of people in the period before the work of Hadamard but that it was not regarded as the last word, perhaps in the way that certain procedures are used in physics now which one knows work very well but where a proper deductive foundation is missing. It is also clear that before Hadamard none of these authors had gone beyond what Riemann had done. In the intervening years, however, the techniques of complex analysis had been developed and were available to a much larger group of mathematicians.

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[29]Carl Ludwig Siegel, über Riemanns Nachlass zur analytischen Zahlentheorie, *Quellen und Studien zur Geschichte der Mathematik* 2 (1932) 45–80, reprinted in [1, pp. 770–805]. Recently Wolfgang Gabcke has re-examined Riemann's papers and reconstructed more of Riemann's arguments.

[30]Hadamard's first paper on the zeta function was submitted for a prize offered by the *Académie des Sciences de Paris* in December, 1892 and published in 1893. For the history of Hadamard's work see V. Maz'ya, T. Shaposhnikova, *Jacques Hadamard, A Universal Mathematician*, AMS/LMS History of Mathematics Series, 14, AMS, 1998, p. 53 ff.

[31]G.B. Mathews, *Theory of Numbers*, Second Edition, Deighton Bell, London, 1892, Sect. 233.

Some attempts were made to go further. Stieltjes made, in 1885, the first unjustified claim to have proved what is now called the Riemann hypothesis which shows that this was already considered to be a significant open question. What was done at this time has been absorbed into the corpus of analytic number theory and we refer to Landau's *Handbuch* for the historical details.

4 Riemann's Mode of Writing

We have seen that many authors have found difficulty in unravelling some of Riemann's arguments. There seem to be a number of reasons for this. Klein writes, rather darkly, of "Anyone who clearly understands under which Riemann worked in Göttingen…".[32] Some of his difficulties are known, for example, the poverty and the illnesses plaguing his family. Klein's formulation indicates, however, that some of the difficulties were associated with his life in Göttingen and here we have only partial information. He clearly had difficult phases but his wife, Elise, said that it was most inaccurate to represent Riemann as a hypochondriac and depressive.[33] His gloomy phases, like those of Eisenstein, seem to have been also to a large extent the result of external factors about which we have relatively little information. These phases, Klein suggests, seem to have made Riemann even slower at writing up his ideas than he might have been under more favourable circumstances.

Leaving this aside there is one aspect that presents itself repeatedly. This is that Riemann does not indicate what mathematics he is assuming. Thus, in his doctoral thesis, he is assuming a knowledge of topology of some sort which he does not seem to feel needs justification. There was at this time just one book on topology, Johann Benedikt Listing's *Vorstudien zur Topologie*. Listing was a physicist alongside Wilhelm Weber in Göttingen. The Physikalisches Kabinett neighboured the university library and so Riemann must have encountered Listing, who, like Weber, seems to have been a gregarious person, often. While Riemann was a student Listing, a polymath of quite remarkable breadth, lectured on such topics as machines, steam engines, and meteorology. He is best known, in appropriate circles, for his work on physiological optics (he wrote the first treatise on this topic). Later he accompanied Sartorius von Waltershausen, Gauss' first biographer, on a vulcanological expedition to carry out measurements of the magnetic field near a volcano. He gave the concept of the equipotential definition of the surface of the Earth, due to Gauss, Stokes, and others, the name "geoid". He also introduced the word "topology" (or rather "Topologie") into mathematics. The phrases *analysis*

[32] See F. Klein, *loc. cit.*, Preface.

[33] Katrin Scheel, *Der Briefwechsel Dedekind-Weber,* de Gruyter, 2014, p. 310.

situs or *geometria situs* was sometimes used in the same context. This phrase had been introduced by Leibniz, but with a different meaning—see Tait's discussion of this point.[34]

In his articles on Listing P.G. Tait[35] has expressed the expectation that there would be much more in Listing's papers on topology than was published in his two books but unfortunately his papers have never been edited.[36]

The relationship between Listing and Riemann is, unfortunately, not reflected in the papers which have been preserved and published but Listing did leave diaries in which his contacts are documented but these have not yet been made available. One can only surmise that Listing was the source of the topology Riemann assumes but even so this topology does seem to go beyond the *Vorstudien*.[37] Listing did not lecture on topology[38] (at least not officially) and we remain rather in the dark as to whether Riemann had direct contact with him in his younger years or whether he had simply studied Listing's book carefully. The anecdote recounted by his teacher Schmalfuss[39] shows how quickly Riemann could absorb and retain mathematics from a written source.

Riemann's laxity in naming his sources seems to be the best way of interpreting an incident which occurred while Riemann was preparing his Habilitationsschrift on trigonometric series. The first section gives an interesting and illuminating survey of the theory up to that point. However, Riemann records here[40] that Dirichlet provided him with a list of the literature which saved him a great deal of work. Riemann was, as one can see from his works, well-read. It would seem that he could easily remember the content but forgot the source. This "help" from Dirichlet may have been a fairly heavy hint that he should mend his ways (and perhaps not adopt Gauss as a role-model).

The problem in reading Riemann is then in finding out what he knew and what he was using. There are two points one should bear in mind. First of all that he belonged to a circle of physicists in Göttingen and that he was very familiar with the literature of mathematical physics. The second point is that one can also gain some information about the literature available to Riemann in Göttingen. The historical collection in the Staats- und Universitätsbibliothek in Göttingen seems to still contain what was available in Riemann's day. On top of the literature, during

[34] P.G. Tait, *Scientific Papers, II,* pp. 85–86.

[35] See P.G. Tait, *loc. cit.*, Papers LXV and LXVI.

[36] The editor of Johann Benedikt Listing, *Beitrag zur physiologischen Optik,* Ostwald's Klassiker, 147, Leipzig 1905, Otto Schwarz has expressed similar views in the area of physiological optics, see p. 51.

[37] This book influenced J. Clerk Maxwell and P.G. Tait who learnt of it from Maxwell, see P.G. Tait, *loc. cit.* p. 86. Its significance is also made clear in the appropriate footnotes in I. Lakatos' *Proofs and Refutations*, Cambridge U.P., 1976.

[38] The lecture lists of this period were published in the *Göttinger Nachrichten* and Listing lectured on such topics as the theory of machines, on steam engines, and on meteorology.

[39] [1, p. 852; also pp. 543, 829–830].

[40] See [1, p. 578].

his student years in Berlin, it seems to have been the lectures of Jacobi and Dirichlet that impressed Riemann most. There is no evidence of any kind of closer contact of Riemann to Jacobi but there is to Dirichlet. Since Dirichlet had over many years close connections to French mathematicians one may assume the Riemann was informed about their work even when this was not directly available to him. In the historical discussion in Riemann's Habilitationsschrift there are many references to the French and Italian literature on the theory of Fourier series[41] presumably from Dirichlet's notes.

One case where this is some interest to us is the long memoir of Binet (1839) on the gamma function. In particular Binet's integral formula for $\log(\Gamma(s))$ gives the complex version of Stirling's formula in a very usable form.[42]

5 The Paper on Prime Numbers

The paper on prime number can be considered as in three parts. The central idea is that there is a duality, given by an integral kernel, between the set of prime numbers and a certain set of complex numbers, the zeros of the zeta function. This is a very general theorem involving techniques applicable in other contexts. It may be seen as in the context of Riemann's work on trigonometric series.

We may see Euler's formula

$$\sin(\pi z) = \pi z \prod_{m \geq 1} (1 - z^2/n^2)$$

as the model for the introduction of the zeros of the zeta function; the ideas needed for the analytic continuation of the zeta function had been around for some time although it was Riemann who clarified the notion of "analytic continuation" in his thesis.[43] With the tools he had available from complex analysis it was much easier to prove the product expansion than it had been for his predecessors, especially those of the eighteenth century.

The Euler product for the zeta function, Euler's proof of the infinitude of primes, and the theory of exponential and trigonometric functions, including the product formula for the sine function all appear in Euler's *Introductio in analysin infinitorum*. The logarithmic derivative of the two sides of the product formula, a little rewritten, gives

[41] Apparently Riemann did not intend to publish this work; it appeared posthumously through the offices of Dedekind a year after Riemann's death. It has, of course, been extremely influential.

[42] See E.T. Whittaker and G.N. Watson, *Modern Analysis,* Cambridge U.P., pp. 248–251.

[43] See A. Weil, On the prehistory of the zeta-function, In: *Number Theory, Trace Formulas and Discrete Groups, Symposium in Honor of Atle Selberg,* Eds. K.E. Aubert, E. Bombieri, D. Goldfeld, Academic Press, 1989, 1–9.

$$\pi \coth(\pi x) = \sum_{m\in\mathbb{Z}} \frac{2x}{x^2 + m^2},$$

a formula which is a consequence of the Poisson Summation Formula. We note, in passing, that the easiest determination of the Fourier transform of $t \mapsto \frac{x}{t^2+x^2}$ is by way of the calculus of residues; another is by way of the formula above. The use of special cases such as this (often in the form of kernels) to study general Fourier and trigonometric series was clear to Riemann, as his Habilitationsschrift shows.

Euler's first attempt at a proof of the product formula for the sine function was to regard the sine function as a generalized polynomial and to equate it with the product over the roots. Euler was, quite rightly, dissatisfied with this argument and in the Introductio gives a convincing proof. Riemann, like all serious analysts of his time, would have been familiar with the Introductio and it would presumably have struck him that Euler's original argument could, in the right circumstances, be rescued. We know now the product theorems of Weierstrass and Hadamard which formalize this idea. These results show that one only needs an estimate on the growth of the function in the region considered and one can avoid the more special properties of the two sides (addition theorem, differential equation) which have been used in other proofs.

From Euler's formula one can develop the theory of the Bernoulli numbers and polynomials. One has with these tools both what one needs to develop the Euler-Maclaurin summation formula and the Fourier expansion of the periodic extension of the Bernoulli polynomials from [0, 1] to $\mathbb{R}$. One approach to this is due to Poisson (1823).[44]

The second part of Riemann's paper, which is a mere sketch, is concerned with the problem of determining the zeros of more precisely. He gives the asymptotic formula and indicates that it is possible to compute the zeros. He even goes further and suggests that "the majority" of the zeros lie on the critical line. At least the second of the two integral representations of the zeta function given in the paper is clearly part of this programme. We know from the Nachlass that Riemann did do some calculations.[45] There is an oral tradition of a "black notebook" of Riemann's

[44]Poisson's argument, where he derives the Euler-Maclaurin formula from the Fourier expansion, is summarized by G.H. Hardy in his book *Divergent Series,* Oxford U.P. 1949, Sect. 13.8. Poisson's original work is "Sur le calcul numérique des Intégrales définies", *Mémoires de L'Académie Royale des Sciences de l'Institut de France,* Année 1823, T. VI, published 1827, 571–602. It is worth remarking that Riemann ignored some of Poisson's real achievements on the theory of Fourier series in the introduction to his Habilitationsschrift. Poisson introduces the Poisson kernel, already an innovation, and uses it to demonstrate that for a continuous function the Fourier series is what Hardy calls summation (A) to the function. Alone, and with the help of Abel's theorem (1826) which states that if $\sum a_n$ converges then $\lim_{x\to 1, x<1} \sum a_n x^n = \sum a_n$, this theorem is sufficient for many application. Dirichlet's theorem, which takes up the notion of a kernel function but one which is much more difficult, is a spectacular piece of analysis but its full force is rarely needed.

[45]H.M. Edwards,*Riemann's Zeta Function,* Academic Press, 1984, pp. 157–158, C.L. Siegel, [1, p. 792], W. Gabcke has been following this up further.

which may or may not still be extant. This tradition seems to come from C.L. Siegel and it is attributed to him in print by H.-J. Klemmt,[46] but the material is extremely fragmentary. There is one page which has been discovered to date in the Nachlass about asymptotic formula. This has been examined by H.-J. Klemmt.[47] According to this Riemann assumed that an asymptotic expansion existed, derived its consequences, and then compared coefficients. Also, according to Klemmt it seems as if Riemann was somewhat uncertain of his ground here. Since one knows how he approached questions in the theory one theta functions one might expect that he used a strategy comprising of the principle of the argument combined with the functional equation and the asymptotics of the gamma function which had been determined by Binet.[48] One also needs an estimation of the argument of the zeta function close to and inside the critical line. The standard argument using Jensen's theorem would have been within Riemann's powers but unfortunately nothing has been found to show that Riemann had taken such an approach. Exactly on which considerations he based his statement about "most" of the zeros being on the line and his formulation of the Riemann Hypothesis remain very obscure. It seems very unlikely that his results went very far. As far as the "black notebook" is concerned, since it was at one time apparently in Landau's possession if it had significant content one would have heard about it.

The final part of Riemann's paper is a very brief discussion of the relation between the explicit formula and the experimental evidence. Riemann claims—without any mention of the nature of the zeros—that the order of magnitude in the error is even $\mathcal{O}(x^{\frac{1}{2}})$. He writes,[49]

[46] H.-J. Klemmt, Asymptotische Entwicklung für kanonische Weierstraßsche Produkte und Riemanns überlegungen zur Nullstellenanzahl der Zetafunktion, *Nachr. Akad. Wiss. Göttingen, II Math.-Phys. Kl.* (1984), Nr. 4, 28–52, esp. p. 47. According to this source the notebook was a collection of formulæ.

[47] *loc. cit.*

[48] See J. Binet, Mémoire sur les intégrales définies eulériennes, *Journal de L'école Polytechnique,* XVI (1839) 240pp. esp. pp. 138 and 144 where Binet's integrals for $\log \Gamma(s)$ are given.

[49] That is, The known approximating expression $F(x) = \mathrm{Li}(x)$ is therefore valid up to quantities of the order $x^{\frac{1}{2}}$ and gives somewhat too large a value; because the non-periodic terms in the expression for $F(x)$ are, apart from quantities that do not grow infinite with x:

$$\mathrm{Li}(x) - \tfrac{1}{2}\mathrm{Li}(x^{\frac{1}{2}}) - \tfrac{1}{3}\mathrm{Li}(x^{\frac{1}{3}}) - \tfrac{1}{5}\mathrm{Li}(x^{\frac{1}{5}}) + \tfrac{1}{6}\mathrm{Li}(x^{\frac{1}{6}}) - \tfrac{1}{7}\mathrm{Li}(x^{\frac{1}{7}}) + \cdots$$

Indeed, in the comparison of $\mathrm{Li}(x)$ with the number of prime numbers less than x, undertaken by *Gauss* and *Goldschmidt* and carried through up to $x =$ three million, this number has shown itself out to be, in the first hundred thousand, always less than $\mathrm{Li}(x)$; in fact the difference grows, with many fluctuations, gradually with x. But also the increase and decrease in the density of the primes from place to place that is dependent on the periodic terms has already excited attention, without, however, any law governing this behaviour having been observed. In any future count it would be interesting to keep track of the influence of the individual periodic terms in the expression for the density of the prime numbers.

Trans. by D.R. Wilkins, Trinity College, Dublin, 1998.

> Die bekannte Näherungsformel $F(x) = \mathrm{Li}(x)$ ist also nur bis auf Grössen von der Ordnung $x^{\frac{1}{2}}$ richtig und giebt einen etwas zu grossen Werth; denn die nicht periodischen Glieder in dem Ausdrucke von $F(x)$ sind, von Grössen, die mit x nicht in's Unendliche wachsen, abgesehen:
>
> $$\mathrm{Li}(x) - \tfrac{1}{2}\mathrm{Li}(x^{\frac{1}{2}}) - \tfrac{1}{3}\mathrm{Li}(x^{\frac{1}{3}}) - \tfrac{1}{5}\mathrm{Li}(x^{\frac{1}{5}}) + \tfrac{1}{6}\mathrm{Li}(x^{\frac{1}{6}}) - \tfrac{1}{7}\mathrm{Li}(x^{\frac{1}{7}}) + \cdots$$
>
> In der That hat sich bei der von *Gauss* und *Goldschmidt* vorgenommenen und bis zu $x =$ drei Millionen fortgesetzten Vergleichung von $\mathrm{Li}(x)$ mit der Anzahl der Primzahlen unter x diese Anzahl schon vom ersten Hunderttausend an stets kleiner als $\mathrm{Li}(x)$ ergeben, und zwar wächst die Differenz unter manchen Schwankungen allmählich mit x. Aber auch die von den periodischen Gliedern abhängige stellenweise Verdichtung und Verdünnung der Primzahlen hat schon bei den Zählungen die Aufmerksamkeit erregt, ohne dass jedoch hierin eine Gesetzmässigkeit bemerkt worden wäre. Bei einer etwaigen neuen Zählung würde es interessant sein, den Einfluss der einzelnen in dem Ausdrucke für die Dichtigkeit der Primzahlen enthaltenen periodischen Glieder zu verfolgen.

One can only conclude that Riemann was so convinced that the Riemann Hypothesis was true that he did not consider the alternative. A little earlier he does mention the other possibility but he seems to forget it. Moreover he seems to expect that his formula is genuinely exact and that it gives a complete answer to the question concerning the distribution of the primes. Calculations show that it is a remarkably good approximation, with the steps at the primes emerging from the continuous approximation but one knows, by a theorem of Littlewood of 1914, that this degree of approximation is not generally true.[50]

One other point in Riemann's paper is interesting. He refers to the calculations of Gauss and *Goldschmidt*, here Benjamin Goldschmidt (1807–1851), Gauss' amanuensis at the Sternwarte. The contribution of Goldschmidt does not seem to be recorded elsewhere.

The date of Gauss' letter to Encke can probably be understood that Gauss had learnt about the recent progress in the study of the primes and wanted to establish his priority without delay. Encke was "his man in Berlin" where Jacobi and Dirichlet were. The intention was presumably that Dirichlet and Jacobi would learn of Gauss' claim. There is no evidence, as far as I know, as to whether this happened, nor do we know how far Gauss' letter was known to others at that time. Gauss seems to have informed others independently. For example, Riemann's reference in his paper to the calculations of Gauss and Goldschmidt, and to Gauss' speculations, seems to indicate a different source as the letter to Encke does not mention Goldschmidt. It is, of course, quite possible that Riemann learnt of this work Goldschmidt and discussed the matter with him but the latter died in 1851.[51] This would seem to be an indication that Riemann got his information directly from someone from the circle of Gauss; if it were not Goldschmidt, Stern, Listing, or Weber would be the most

[50] See A.E. Ingham, *The Distribution of Prime Numbers,* Cambridge U.P., 1932, p. 105.

[51] In fact Riemann hoped to taken on his position in the Sternwarte but Gauss apparently recognized that he was destined for higher goals—s. [1, p. 578]. Riemann attended a lecture course by Goldschmidt on terrestrial magnetism in 1846—s. [1, p. 575].

likely sources. Amongst these Stern was the one who had the strongest interest in number theory and Dedekind records that Riemann attended several lectures of his. It was Stern who befriended Gotthold Eisenstein after Eisenstein visited Göttingen in 1844 and to whom Eisenstein sent a number of very personal letters; Stern seems to have been a surrogate father to Eisenstein who seems to have broken with his own family at an early age—he seems to have had a relationship with his father not unlike that of Franz Kafka with his. It was also Stern who made possible the contact between Eisenstein and Gauss.[52] The positive impression one is left of Stern is confirmed by Rudio's graveside oration at Stern's funeral in Zürich.[53]

It is not clear if the papers of Chebyshev would have been available to Riemann in journal form—they do not seem to have been in the library in Göttingen—but they were known to Dirichlet so we can assume that Riemann learnt of their content from him. There is no evidence that he had any direct contact with Gauss in connection with prime numbers but Gauss does seem to have been informed about him and seems to have followed his progress with interest. One may surmise that Riemann learnt of Gauss' work on prime numbers partly through Dirichlet, and partly through intermediaries in Göttingen. The letter to Encke seems only to have become generally known after it was reprinted in Volume 2 of Gauss' *Werke* in 1863. This volume appeared while Riemann was still alive, a few years after he wrote his paper on prime numbers. It seems to have been edited by Dedekind, Stern, and Schering; we know that both Dedekind and Schering were moderately close to Riemann and one may suspect that the same is true of Stern. All three were members of the *Königliche Akademie der Wissenschaften zu Göttingen* as was Riemann,[54] which was, as a body, responsible for editing and publishing Gauss' *Werke*. It is striking that there is nothing in the "Bemerkungen" on Gauss' letter either by Riemann or referring to his paper. That Riemann, although he was a member of the Königliche Akademie was not directly involved with this issue is presumably because he had suffered a collapse in his health in 1862 and was in Italy for a long period to recover. That there is no reference to Riemann's paper is striking.

6 The Zeta Function and Bessel Functions

The calculation of the real zeros of the ξ function, the zeros of the zeta function on the critical line, was not a question that caused too much difficulty at that time, either in principle or in practice. It seems to have been known that the integral representation given by Riemann in the form

[52] See G. Eisenstein, *Mathematische Werke, Bd. 2,* Chelsea, New York, 1975, 898–904.

[53] See F. Rudio, Erinnerungen an M.A. Stern, *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich,* v. 39–40 (1894–1895) 131–143. Note that Rudio stresses the friendship between Stern and Henle.

[54] He had been elected in 1859.

$$\xi(t) = 4\int_1^\infty \frac{\mathrm{d}(x^{\frac{3}{2}}\psi'(x))}{\mathrm{d}x} x^{-\frac{1}{4}} \cos\left(\frac{1}{2}t\log x\right)\mathrm{d}x$$

is closely analogous to integral representations of Bessel functions. This was made explicit by G. Pólya in 1926[55] but there are implicit indications that it was recognized much earlier. Thus an expert on the calculation of Bessel functions, E. Meissel, made some rough calculations of the coefficients of the ξ function around $t = 0$ and could use this for rough, non-rigorous estimates of a few zeros. E. Meissel was an expert numerical mathematician who also constructed particularly accurate tables, published in 1888, of the Bessel functions J_0 and J_1. The zeros of these functions had appeared in a number of investigations before 1850. Riemann uses the Bessel functions I_0 and K_0 in his paper of 1855 on Nobili rings.[56]

The zeros of the J_0 appear in the Fourier-Bessel expansion which was introduced by Fourier in his *Théorie de chaleur* (1822) and the zeros appear in several other investigations which were undertaken in the central part of the nineteenth century. Stimulated by Airy's work on optics G.G. Stokes developed a particularly effective method of computing the zeros of various Bessel functions (1850).

Although Riemann does not quote Stokes[57] he makes use, in his paper on Nobili rings, of "halbconvergenten Reihen"[58] which are those established by Stokes. It seems then quite probable that Riemann knew Stokes' paper. We know that the volume of *Transactions of the Cambridge Philosophical Society* where it appeared was available in Göttingen; I am indebted to J. Mangei and H. Rohlfing of the Handschriftenabteilung/Historische Bibliothek of the SUB Göttingen for this information. Moreover Riemann indicates familiarity with other papers of Stokes in connection with the theory of shock waves.[59] A transformation that Riemann applies in his paper on Nobili rings, pp. 89–90, and commented by the editors, is the same as one used by Stokes. Stokes does not use the methods of function theory and so his proof is in the context of real variables but is on the threshold of complex function theory. Stokes uses it in his paper to convert an integral introduced in 1838 by the Astronomer Royal, G.B. Airy in the course of certain optical investigations

$$\int_0^\infty \cos\left(\frac{\pi}{2}(w^3 - xw)\right)\mathrm{d}w$$

into a rapidly convergent integral

[55] G. Pólya, Bemerkung über die Integraldarstellung der Riemannschen ξ-Funktion, *Acta Math.* 48 (1926) 305–317.

[56] See [1, pp. 87–98].

[57] G.G. Stokes, On the numerical calculation of a class of definite integrals and infinite series, in *Collected Works, Vol. 2,* 329–357.

[58] See [1, p. 91].

[59] [1, p. 208].

$$\frac{1}{2}\int_0^\infty e^{\pi i/6}\exp(-u^3+xue^{2\pi i/3})+e^{-\pi i/6}\exp(-u^3+xue^{-2\pi i/3})\mathrm{d}u$$

from which he can determine the differential equation which it satisfies. He uses this first of all to find a differential equation, not too different from the one that Airy's function satisfies and from this, by iterating a geometrical comparison method finds a "semi-convergent" series which gives a very good approximation for the Airy function and which also gives a very effective method for computing the zeros. He emphasizes just how much easier his method is than Airy's and gives tables. Stokes' method also shows, in a later notation, the connection between Airy's function and Bessel functions of order $\frac{1}{3}$ (theorem of Wirtinger (1897) and Nicholson(1909)). Stokes gives examples of further examples of his method applied to other Bessel functions. Also in this paper Stokes makes use, probably for the first time, of the principle of stationary phase.[60] This is usually attributed to W. Thomson in a paper of 1887.[61]

There is a broad discussion of the history of the calculation of the zeros of Bessel functions in G.N.Watson, *A Treatise on the Theory of Bessel functions*, Ch.VII. Oddly in this discussion Watson does not discuss Stokes' method. Most of the methods he does look at use one or another integral representation. Watson does mention Stokes integral representation of the Airy function in Sect. 6.4 (from which Stokes derived the differential equation) and Stokes' computation of the zeros of the Airy function and of J_0 and J_1 (Sects. 15.52 and 15.53).

We have indicated above that the zeros of the Bessel function J_0 appear in the Fourier-Bessel expansion.[62] This is an expansion similar in form to Riemann's explicit formula. What is particularly interesting here is that the analogue of the Riemann Hypothesis is actually valid as was fist proved by Fourier himself. Fourier's proof was reworked by Hurwitz in 1889.[63] There is a second proof, attributed by Watson to Lommel[64] but which was also, apparently independently, discovered by Riemann and published at about the same time as Lommel's proof by K. Hattendorf in his version of Riemann's lectures on the partial differential equations of mathematical physics.[65] Fourier's proof involves rational approximations to J_0, derived from the differential equation, to which one can apply the methods for studying the zeros of polynomials; the Lommel-Riemann proof involves an integral equation which is also derived from the differential equation and one might, with

[60] *loc. cit.* p. 340. It is perhaps worth noting that Stokes' idea was an extension, to an oscillating integral, of a method used by Laplace (1820) for non-oscillating integrals which have an exponential kernel; see E.T.Copson, *Asymptotic Expansions,* Cambridge U.P. 1965, Sect. 17.

[61] See the remarks in H. Lamb, *Hydrodynamics*, Dover, 1945, p. 395.

[62] See G.N. Watson, *loc. cit.*, Chap. XVIII.

[63] For the details see Watson, *loc. cit.,* Sect. 15.27.

[64] See Watson, *loc. cit.,* Sect. 15.25.

[65] See K.Hattendorf, *Partielle Differentialgleichungen und deren Anwendung auf physicalische Fragen. Vorlesungen von Bernhard Riemann,* Braunschweig, 1869, Sect. 97.

some license, see it as a precursor of a proof in the realm of spectral theory in Hilbert space. The zeros of J_0 are very regular in a way that the zeros of the Riemann zeta function are not. In particular Gram's law describing the distribution of the zeros of the zeta function fails badly. This seems to rule out any approach to the Riemann Hypothesis along the lines of the Fourier proof; whether Hilbert's expectation that a spectral approach could be successful cannot be ruled out but over a century of failed attempts counsels caution.

In view of these considerations one can view the theory of the ξ function as analogous that part of the theory of Bessel functions that one can establish without the use of the differential equation (or some related property). The first step in Stokes' paper of using, in effect, Cauchy's theorem and the step using the method of stationary phase belong to this category. Riemann goes much further—Wolfgang Gabcke has informed me that Riemann actually uses the Sattelpunkt method in his notes. It was only much later that this became a standard technique and it is usually ascribed to P. Debye.[66] However, it developed during the second half of the nineteenth century and Debye gives historical details. The idea, as he points out, began with a fragment of Riemann's, written in Italian, dated October, 1863 and edited by H.A. Schwarz.[67]

7 Changes in Our Understanding of Riemann

In recent times there have been new documents published which help to deepen our understanding of Riemann's thinking generally. In particular the Dedekind-Weber correspondence[68] throws a great deal of light on Riemann's life and his associations with those around him. In particular his wife Elise (*née* Koch) appears in a much more active and supportive role than had been visible before. Besides this correspondence some letters to Wilhelm Weber were discovered recently by J.M. Marín, and published by W. Gabcke.[69] In particular a letter of 18th April, 1866, that is just 3 months before his death, show that Riemann was actively involved with the theory and physiology of the ear, more so than the fragment published in the Collected Papers[70] would suggest. He was dissecting the ears of slaughtered animals

[66] P. Debye, Näherungsformel für die Zylinderfunktionen für große Werte des Arguments und unbeschränkt veränderliche Werte des Index, *Math.Ann.* 67 (1909) 535–558.

[67] [1, pp. 456–462]. For more information see E.B. Barnes, The Asymptotic Expansion of Integral Functions defined by Taylor's Series, *Phil. Trans. Royal Soc. London. Series A,* Vol. 206 (1906), pp. 249–297, and A Memoir on Integral Functions, *Phil. Trans. Royal Soc. London. Series A,* Vol. 199 (1902), pp. 411–500.

[68] Katrin Scheel, Der Briefwechsel Dedekind-Weber, de Gruyter, 2014.

[69] W. Gabcke, Transcriptions of six letters of Bernhard Riemann preserved in the SIL, *arXiv*:1602.08264 (2016).

[70] [1, p. 370ff.]. In a footnote by E. Schering and J. Henle it is remarked that Helmholtz' work was a stimulus for Riemann's interest. Jakob Henle (1809–1895), probably not a name known to most

in order to develop a mathematical theory of the ear. Perhaps he was inspired to this endeavour by Listing's work on physiological optics. That he was involved in such very physical investigations opens questions as to exactly what state his health was in. His final journey to Selasca was apparently a flight from the war that was breaking out between Prussia and Hannover; the situation in Göttingen was corresponding chaotic.[71] Dedekind in the Lebenslauf indicates very discretely that the journey was very strenuous. Both Archimedes and Kepler died through being caught up in wars and it seems as if the same may well be true of Riemann. This is a sobering thought at times like this when so many are in similar situations.

In the article by H. Freudenthal quoted above that author puts forward the view that Riemann would have tended towards philosophy had he lived longer. These more recent finds make this unlikely if one understands "philosophy" in the sense of someone like Kant; if one understands the word in an old-fashioned sense as "Natural Philosophy", then the remark would be justified.

It is remarkable that although Riemann had the greater number of contacts with physicists, mathematical and otherwise, the works for which he is most remembered belong to what we would consider today to be pure mathematics. It is worth comparing Riemann with two of his near contemporaries G.G. Stokes (1819–1903) and W. Thomson (1824–1907) both of whom were very interested, and contributed substantially to, recent mathematical developments of their time, but in later years worked on very broad ranges of subjects. Hermann v. Helmholtz (1821–1894) had a similarly broad range on interests; in his case his mathematical interests developed rather later and in fact took up the question as to the nature of geometry about which Riemann had thought deeply. The indications are that Riemann would have been a kindred spirit of these scholars—but it is perhaps better not to speculate too far. It is, though, worthy of note that according to a footnote[72] by E. Schering and J. Henle, an anatomist in Göttingen, Riemann had been inspired to study the ear by reading the works of Helmholtz. It is perhaps worth recalling that it is recorded that "Riemann sang like a canary".[73]

In quite recent times documents of Riemann's and those with some sort of connection with him have been found and shed new light on Riemann's life and work; one may assume that there are more that still many more unregarded in various archives. One knows little about Riemann's life, for example, if Klein's

mathematicians, was a physiologist who worked from 1852 in Göttingen. He worked on many topics and was especially a gifted microscopist who studied the cochlea. Riemann was interested in studying the cochlea of slaughtered animals but in the letter in question explains that he had great difficulty in extracting the delicate cochlea from the bone in which it is embedded. For the historical context of Riemann's letter see the second footnote on p. 145 of H. Helmholtz, *On the sensations of tone,* Dover, 1954. It is perhaps of interest to note that it was Henle to whom Dedekind submitted his draft of the *Lebenslauf* for comments; see. K. Scheel *loc. cit.*, p. 112.

[71] See K. Wettig, Als die Hannoveraner Preußen wurden, *Göttinger Tageblatt,* 20. Juni 2016, p. 16.

[72] [1, p. 370]; see also [1, pp. 843–844].

[73] This was reported to (the notoriously unmusical) F. Klein by M. Stern—see F. Klein, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, Bd. I, p. 249.

remark does refer to aspects of his life in Göttingen in the middle 1850s which are not yet in the literature. Klein's remark suggests that it would illuminate our understanding of Riemann's development. Riemann does seem to have been much happier in Italy. We know very little about his relationships to Wilhelm Weber and Johann Listing. Of particular interest in this would be the papers of J.B. Listing.

Reference

1. B. Riemann, *Gesammelte Mathematische Werke, Wissenschaftlicher Nachlass und Nachträge*, Collected Papers, ed. by H. Weber, R. Dedekind, R. Narashiman (Springer, Berlin, 1990)

A Taniyama Product for the Riemann Zeta Function

David E. Rohrlich

Abstract A Taniyama product for the Riemann zeta function is defined and an analogue of Mertens' theorem is proved.

1 Introduction

Tucked unobtrusively into Taniyama's memoir [4] on compatible families of ℓ-adic representations is a curious identity expressing the zeta function of such a family as an infinite product of imprimitive Artin L-functions [4, p. 356, Theorem 3]. The simplest case of the identity (arising from the cyclotomic character, or from its inverse, depending on one's conventions) is

$$\zeta(s-1)/\zeta(s) = \prod_{c\geqslant 1} \zeta_c(s), \tag{1}$$

where $\zeta(s)$ is the Riemann zeta function and $\zeta_c(s)$ is the imprimitive Dedekind zeta function—imprimitive because the Euler factors at the primes dividing c are removed—of the cyclotomic field K_c generated over $\mathbb{Q}$ by the cth roots of unity. Thus $\zeta_1(s)$ is $\zeta(s)$ itself, $\zeta_2(s)$ is $(1-2^{-s})\zeta(s)$, and so on. The infinite product converges for $\Re(s) > 2$.

In this note we modify (1) slightly so as to obtain a product for $\zeta(s)$ rather than $\zeta(s-1)/\zeta(s)$. Let K_c^+ be the maximal totally real subfield of K_c, and let $\zeta_c^+(s)$ be the Dedekind zeta function of K_c^+ with the Euler factors at the primes dividing c removed. Then

$$\zeta(s) = \prod_{c\geqslant 1} \zeta_c^+(s+1). \tag{2}$$

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H. Montgomery et al. (eds.), *Exploring the Riemann Zeta Function*,
DOI 10.1007/978-3-319-59969-4_12

Like the traditional Euler product

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \tag{3}$$

the Taniyama product (2) converges for $\Re(s) > 1$.

The main result of this note can be viewed as an analogue of Mertens' theorem [2]. It bears the same relation to (2) as Mertens' theorem does to (3), and Mertens' theorem itself figures prominently in the proof. Let γ denote the Euler–Mascheroni constant.

Theorem 1 $\prod_{c \leqslant x} \zeta_c^+(2) \sim e^\gamma \log x$.

Of course $\zeta_c^+(2)$ can be computed explicitly in terms of generalized Bernoulli numbers. For a primitive Dirichlet character χ of conductor q, let

$$b_{2,\chi} = \sum_{j=1}^{q} \chi(j)(j^2/q - j + q/6).$$

Also write d_c^+ for the discriminant of K_c^+, and let $\varphi(c)$ be the cardinality of $(\mathbb{Z}/c\mathbb{Z})^\times$. For the sake of a succinct formula we put

$$\phi(c) = \begin{cases} \varphi(c) & \text{if } c \geqslant 3 \\ 2 & \text{if } c = 1 \text{ or } 2. \end{cases}$$

Then

$$\zeta_c^+(2) = \frac{\pi^{\phi(c)}}{{d_c^+}^{3/2}} \prod_{q|c} \sideset{}{^*}\prod_{\substack{\chi \bmod q \\ \chi(-1)=1}} b_{2,\chi} \prod_{p|c} (1 - \chi(p)p^{-2}), \tag{4}$$

where the asterisk indicates that in the second product, χ runs over *primitive* characters of conductor q.

Expressions similar to (4) have arisen in other contexts. For example, nearly the same triple product occurs in a formula of Yu [6] for the order of a certain cuspidal divisor class group of the modular curve $X_1(N)$ (see also Yang [5, p. 521]). Even so, the differences between Yu's formula and (4) appear to be significant enough to preclude a straightforward interpretation of Theorem 1 as an asymptotic average of cuspidal divisor class numbers.

It is a pleasure to thank the referee for a careful reading of the manuscript.

2 Taniyama's Identity

A proof of (1) is of course subsumed in Taniyama's proof of his general formula, but we will nonetheless sketch a proof here before going on to the modification (2). For a prime p not dividing c let $f(p,c)$ be the order of the residue class of p in $(\mathbb{Z}/c\mathbb{Z})^\times$. Also, write $Z(s)$ for the right-hand side of (1). Then $Z(s)$ can be written as the double product

$$Z(s) = \prod_{c\geqslant 1}\prod_{p\nmid c}(1-p^{-sf(p,c)})^{-\varphi(c)/f(p,c)}, \tag{5}$$

where the inner product is $\zeta_c(s)$ and runs over primes not dividing c. The proof of (1) amounts to reversing the order of multiplication in the double product in (5). By choosing a branch of $\log Z(s)$ we can do the computation additively, and the absolute convergence of the resulting triple sum in the right half-plane $\Re(s) > 2$ will show a posteriori that the original double product is meaningful in this region and that the calculation is legitimate.

We define our branch of $\log Z(s)$ by

$$\log Z(s) = \sum_{c\geqslant 1}\sum_{p\nmid c}\frac{\varphi(c)}{f(p,c)}\sum_{m\geqslant 1}\frac{p^{-mf(p,c)s}}{m}. \tag{6}$$

Putting $d = mf(p,c)$ and summing over $d \geqslant 1$, we obtain

$$\log Z(s) = \sum_{c\geqslant 1}\sum_{d\geqslant 1}\sum_{\substack{p\nmid c\\ f(p,c)|d}}\frac{\varphi(c)}{d}p^{-ds}$$

or equivalently (since $f(p,c)|d$ if and only if $p^d \equiv 1 \bmod c$)

$$\log Z(s) = \sum_{p}\sum_{d\geqslant 1}\sum_{\substack{c\geqslant 1\\ c|p^d-1}}\frac{\varphi(c)}{d}p^{-ds}.$$

As $\sum_{m|n}\varphi(m) = n$, we conclude that

$$\log Z(s) = \sum_{p}\sum_{d\geqslant 1} d^{-1}p^{-ds}(p^d-1).$$

The inner sum equals $\log((1-p^{1-s})^{-1}(1-p^{-s}))$, and (1) follows.

3 The Modification

The proof of (2) is much the same. Put $\varphi^+(c) = [K_c^+ : \mathbb{Q}]$, so that $\varphi^+(c)$ is 1 if $c = 1$ or 2 and $\varphi(c)/2$ otherwise. (Note also that $\varphi^+(c) = \phi(c)/2$, where $\phi(c)$ is as in the introduction.) If p is a prime not dividing c, then the order in $\mathrm{Gal}(K_c^+/\mathbb{Q})$ of a Frobenius at p will be denoted $f^+(p,c)$. Of course $f^+(p,c)$ is also the order of the coset represented by p in the quotient of $(\mathbb{Z}/c\mathbb{Z})^\times$ by the image of $\{\pm 1\}$. Let $Z^+(s)$ denote the right-hand side of (2), and write $Z^+(s)$ as a double product:

$$Z^+(s) = \prod_{c\geqslant 1}\prod_{p\nmid c}(1-p^{-sf^+(p,c)})^{-\varphi^+(c)/f^+(p,c)}. \tag{7}$$

Define a branch of $\log Z^+(s)$ by setting

$$\log Z^+(s) = \sum_{c\geqslant 1}\sum_{p\nmid c}\frac{\varphi^+(c)}{f^+(p,c)}\sum_{m\geqslant 1}\frac{p^{-mf^+(p,c)s}}{m}. \tag{8}$$

The calculation will again show that the triple sum is absolutely convergent for $\Re(s) > 2$. Putting $d = mf^+(p,c)$ and summing over $d \geqslant 1$, we find

$$\log Z^+(s) = \sum_{c\geqslant 1}\sum_{d\geqslant 1}\sum_{\substack{p\nmid c\\ f^+(p,c)|d}}\frac{\varphi^+(c)}{d}p^{-ds}$$

as before. But the condition $f^+(p,c)|d$ means $c|p^d-1$ or $c|p^d+1$, so we get

$$\log Z^+(s) = \sum_{p}\sum_{d\geqslant 1}\sum_{c|(p^d\pm 1)}\frac{\varphi^+(c)}{d}p^{-ds}. \tag{9}$$

We emphasize that the innermost sum is the sum over all c such that at least one of the conditions $c|p^d-1$ and $c|p^d+1$ is satisfied.

If $c \geqslant 3$, then the conditions $c|p^d-1$ and $c|p^d+1$ are mutually exclusive and $\varphi^+(c) = \varphi(c)/2$, so we have

$$\sum_{\substack{c\geqslant 3\\ c|p^d\pm 1}}\varphi^+(c) = \frac{1}{2}\sum_{\substack{c\geqslant 3\\ c|p^d-1}}\varphi(c) + \frac{1}{2}\sum_{\substack{c\geqslant 3\\ c|p^d+1}}\varphi(c). \tag{10}$$

On the other hand, if $c = 1$ or 2, then the conditions $c|p^d-1$ and $c|p^d+1$ are both satisfied (for if $c = 2$, then p is odd), but $\varphi^+(c) = 1$. So Eq. (10) is correct without the restriction $c \geqslant 3$, and the identity $\sum_{m|n}\varphi(m) = n$ gives

$$\sum_{c|(p^d\pm 1)}\varphi^+(c) = \frac{1}{2}((p^d-1)+(p^d+1)) = p^d. \tag{11}$$

Multiplying through by p^{-ds}/d in (11) and inserting the result in (9), we obtain

$$\log Z^+(s) = \sum_p \log(1 - p^{1-s})^{-1},$$

or in other words

$$\prod_{c \geqslant 1} \zeta_c^+(s) = \zeta(s-1) \tag{12}$$

for $\Re(s) > 2$. Replacing s by $s+1$ gives (2) for $\Re(s) > 1$.

4 The Analogue of Mertens' Theorem

We shall prove that

$$\prod_{c \leqslant x+1} \zeta_c^+(2) \sim e^\gamma \log x. \tag{13}$$

Theorem 1 is an immediate consequence of (13), because $\log(x-1) \sim \log x$.

We proceed as in the derivation of (2), but with two crucial changes: first, we take $s = 2$, and second, c now runs over the *finite* set of positive integers $\leqslant x+1$. Thus (8) is replaced by

$$\log \prod_{c \leqslant x+1} \zeta_c^+(2) = \sum_{c \leqslant x+1} \sum_{p \nmid c} \frac{\varphi^+(c)}{f^+(p,c)} \sum_{m \geqslant 1} \frac{p^{-2mf^+(p,c)}}{m}. \tag{14}$$

Next we make the change of variables $d = mf^+(p,c)$. Since c runs over a finite set and the Dirichlet series for $\log \zeta_c^+(s)$ is absolutely convergent for $\Re(s) > 1$ and in particular for $s = 2$, we can rearrange the order of summation to obtain

$$\log \prod_{c \leqslant x+1} \zeta_c^+(2) = \sum_p \sum_{d \geqslant 1} \sum_{\substack{c \leqslant x+1 \\ c \mid p^d \pm 1}} \varphi^+(c) \frac{p^{-2d}}{d} \tag{15}$$

as in (9). For the sake of notational simplicity, we conflate the double sum over p and d into a single sum over p^d, and we put

$$\Phi(p^d, x) = \frac{p^{-2d}}{d} \sum_{\substack{c \leqslant x+1 \\ c \mid p^d \pm 1}} \varphi^+(c). \tag{16}$$

Then (15) can be written in the form

$$\log \prod_{c\leqslant x+1} \zeta_c^+(2) = \sum_{\substack{p^d>x\\ d\geqslant 2}} \Phi(p^d,x) + \sum_{p>x} \Phi(p,x) + \sum_{p^d\leqslant x} \Phi(p^d,x). \tag{17}$$

We shall prove the following assertions:

$$\sum_{\substack{p^d>x\\ d\geqslant 2}} \Phi(p^d,x) = o(1). \tag{18}$$

$$\sum_{p>x} \Phi(p,x) = o(1). \tag{19}$$

$$\sum_{p^d\leqslant x} \Phi(p^d,x) = \log \prod_{p\leqslant x} (1-p^{-1})^{-1} + o(1). \tag{20}$$

Granting these statements and using them in (17), we find that

$$\log \prod_{c\leqslant x+1} \zeta_c^+(2) = \log \prod_{p\leqslant x} (1-p^{-1})^{-1} + o(1),$$

whence exponentiation and an appeal to Mertens' theorem give (13).

To prove (18), we first note that

$$\sum_{\substack{c\leqslant x+1\\ c\mid p^d\pm 1}} \varphi^+(c) \leqslant p^d. \tag{21}$$

by (11). Thus $\Phi(p^d,x) \leqslant p^{-d}/d$ by (16), whence the left-hand side of (18) is bounded by the sum of the terms with $p^d > x$ in the convergent double series $\sum_p \sum_{d\geqslant 2} p^{-d}/d$. Since the tail of a convergent series is $o(1)$, we obtain (18).

Next we prove (19). Take $x > 20$. It suffices to show that the sums

$$\sum\nolimits_1 = \sum_{x<p\leqslant x\log x} \Phi(p,x)$$

and

$$\sum\nolimits_2 = \sum_{p>x\log x} \Phi(p,x)$$

are both $o(1)$. Appealing once again to (21) and (16), we see that

$$\sum\nolimits_1 \leqslant \sum_{x<p\leqslant x\log x} p^{-1} = \log(\log(x\log x)/(\log x)) + o(1)$$

(cf. Chebyshev [1]). But $\log(\log(x\log x)/(\log x)) = \log(1+o(1))$, which is $o(1)$. Thus the sum $\sum_1$ is $o(1)$.

For $\sum_2$ we revert to an earlier order of summation:

$$\sum\nolimits_2 = \sum_{c\leqslant x+1} \varphi^+(c) \sum_{\substack{p\equiv \pm 1 \bmod c \\ p>x\log x}} p^{-2}. \tag{22}$$

We then rewrite the inner sum using Abel summation:

$$\sum_{\substack{p\equiv \pm 1 \bmod c \\ p>x\log x}} p^{-2} = \frac{\pi(y;c,\pm 1)}{y^2}\Bigg|_{x\log x}^{\infty} + 2\int_{x\log x}^{\infty} \frac{\pi(y;c,\pm 1)}{y^3}\,dy, \tag{23}$$

where $\pi(y;c,\pm 1)$ is the number of primes $\leqslant y$ congruent to $\pm 1 \bmod c$. By the strong form of the Brun–Titchmarsh theorem due to Montgomery and Vaughan [3], we have

$$\pi(y;c,\pm 1) \leqslant \frac{4y}{\varphi(c)\log(y/c)}. \tag{24}$$

Using (24) in (23) and then inserting the result in (22), we see that

$$\sum\nolimits_2 \leqslant \sum_{c\leqslant x+1} 8\int_{x\log x}^{\infty} \frac{dy}{y^2\log(y/c)},$$

the term $-\pi(x\log x;c,\pm 1)/(x\log x)^2$ having simply been omitted since it is negative. For $x > 20$ we have $(x\log x)/c > e$; hence the integrand is $\leqslant y^{-2}$ and the integral is $\leqslant (x\log x)^{-1}$. It follows that the sum over c is $\leqslant (1+1/x)/(\log x)$ and thus $o(1)$.

Finally we prove (20). The summation on the left-hand side of (20) is restricted to $p^d \leqslant x$, so if $c|p^d \pm 1$, then $c \leqslant x+1$. Hence $\Phi(p^d,x)$ coincides with p^{-d}/d by (11) and (16), so

$$\sum_{p^d\leqslant x} \Phi(p^d,x) = \sum_{p^d\leqslant x} p^{-d}/d.$$

We may write this identity as

$$\sum_{p^d\leqslant x} \Phi(p^d,x) = \sum_{p\leqslant x} \sum_{d\leqslant \frac{\log x}{\log p}} p^{-d}/d, \tag{25}$$

while

$$\log \prod_{p \leqslant x} (1 - p^{-1})^{-1} = \sum_{p \leqslant x} \sum_{d \geqslant 1} p^{-d}/d. \tag{26}$$

Subtracting (25) from (26), we see that

$$\log \prod_{p \leqslant x} (1 - p^{-1})^{-1} - \sum_{p^d \leqslant x} \Phi(p^d, x) = \sum_{p \leqslant x} \sum_{p^d > x} p^{-d}/d \tag{27}$$

If $p \leqslant x$ and $p^d > x$, then $d \geqslant 2$, so (27) gives

$$\log \prod_{p \leqslant x} (1 - p^{-1})^{-1} - \sum_{p^d \leqslant x} \Phi(p^d, x) \leqslant \sum_{\substack{p^d > x \\ d \geqslant 2}} p^{-d}/d.$$

The left-hand side is positive by (27), and as noted previously, the right-hand side is the tail of a convergent double series, and therefore $o(1)$. Hence the left-hand side is $o(1)$, and (20) follows.

5 The Special Value

For the sake of completeness, we recall the standard calculation of $\zeta_c^+(2)$ in terms of generalized Bernoulli numbers. Write $\zeta_c^+(s)$ as a product of Dirichlet L-functions associated with even Dirichlet characters to the modulus c:

$$\zeta_c^+(s) = \prod_{\substack{\chi \bmod c \\ \chi(-1)=1}} L(s, \chi). \tag{28}$$

We restrict attention to primitive characters by writing

$$\zeta_c^+(s) = \prod_{q|c} {\prod_{\substack{\chi \bmod q \\ \chi(-1)=1}}}^{*} L(s, \chi) \prod_{p|c} (1 - \chi(p) p^{-s}). \tag{29}$$

Now recall the functional equation of $L(s, \chi)$: For χ even and primitive of conductor q, let

$$\Lambda(s, \chi) = q^{s/2} \pi^{-s/2} \Gamma(s/2) L(s, \chi); \tag{30}$$

then

$$\Lambda(s, \chi) = W(\chi) \Lambda(1 - s, \overline{\chi}), \tag{31}$$

where $W(\chi)$ is the root number of χ. On the other hand, according to a classic formula we have $L(1-k,\chi) = -b_{k,\chi}/k$ for integers $k \geqslant 2$ (and actually even for $k = 1$ if $\chi \neq 1$). Taking $k = 2$ and applying (31), we obtain

$$L(2,\chi) = \pi^2 b_{2,\overline{\chi}} W(\chi)/q^{3/2}. \tag{32}$$

Next recall that if χ has order $\geqslant 3$ then $W(\chi)W(\overline{\chi}) = 1$, while if $\chi^2 = 1$, then $W(\chi) = 1$. Thus on substituting (32) in (29), we obtain

$$\zeta_c^+(2) = \prod_{q|c} (\pi^2/q^{3/2})^{\psi^+(q)} \sideset{}{^*}\prod_{\substack{\chi \bmod q \\ \chi(-1)=1}} b_{2,\chi} \prod_{p|c} (1-\chi(p)p^{-2}), \tag{33}$$

where $\psi^+(q)$ is the number of even Dirichlet characters which are primitive of conductor q. Since $\sum_{q|c} \psi^+(q) = \varphi^+(c)$ we have

$$\prod_{q|c} (\pi^2)^{\psi^+(q)} = \pi^{\phi(c)}. \tag{34}$$

Furthermore

$$\prod_{q|c} q^{\psi^+(q)} = d_c^+, \tag{35}$$

as one sees, for example, by observing that the exponential factor in the functional equation of the Dedekind zeta function of K_c^+ is $(d_c^+)^{s/2}$, while the exponential factor in (31) or rather (30) is $q^{s/2}$. On substituting (34) and (35) in (33), we obtain (4).

6 A Question

For $c \geqslant 3$, let ξ_c be the quadratic Hecke character of K_c^+ associated with the extension K_c/K_c^+, and let $L(s,\xi_c)$ be the corresponding Hecke L-function. Write $L_c(s)$ for the imprimitive Hecke L-function obtained from $L(s,\xi_c)$ by deleting the Euler factors at the primes dividing c. Also put $L_c(s) = 1$ for $c = 1$ or 2. Then $L_c(s) = \zeta_c(s)/\zeta_c^+(s)$ in all cases. Hence combining (1) with (12), we obtain

$$\zeta(s)^{-1} = \prod_{c \geqslant 1} L_c(s).$$

The infinite product converges for $\Re(s) > 2$, but $1/\zeta(s)$ is holomorphic and nonvanishing for $\Re(s) > 1$. Is the true region of convergence perhaps much larger than $\Re(s) > 2$? We can offer only a minimal enlargement:

Theorem 2 *The product* $\prod_{c\geqslant 1} L_c(s)$ *converges to* $\zeta(s)^{-1}$ *for* $\Re(s) \geqslant 2$.

Proof For integers $c \geqslant 3$ and primes $p \nmid c$, put $\kappa(p,c) = \xi_c(\mathfrak{p})$, where $\mathfrak{p}$ is a prime ideal of K_c^+ lying above p. If $c = 1$ or 2 put $\kappa(p,c) = 0$. Then

$$L_c(s) = \prod_{p\nmid c} (1 - \kappa(p,c)p^{-f^+(p,c)s})^{-\varphi^+(p,c)/f^+(p,c)}$$

for $\Re(s) > 1$, and consequently

$$\log \prod_{c\leqslant x+1} L_c(s) = \sum_{c\leqslant x+1} \sum_{p\nmid c} \frac{\varphi^+(c)}{f^+(p,c)} \sum_{m\geqslant 1} \kappa(p,c)^m \frac{p^{-mf^+(p,c)s}}{m}.$$

Making the change of variables $d = mf^+(p,c)$ as before, we obtain

$$\log \prod_{c\leqslant x+1} L_c^+(s) = \sum_p \sum_{d\geqslant 1} \sum_{\substack{c\leqslant x+1 \\ c|p^d\pm 1}} \varphi^+(c)\,\kappa(p,c)^{d/f^+(p,c)} \frac{p^{-ds}}{d}$$

(note that the condition $c|p^d \pm 1$ means precisely that $f^+(p,c)|d$). All of this is valid for $\Re(s) > 1$, but we now assume that $\Re(s) \geqslant 2$ or simply that $\Re(s) = 2$, since the case $\Re(s) > 2$ has already been dealt with. Put

$$\Psi(p^d, x, s) = \frac{p^{-ds}}{d} \sum_{\substack{c\leqslant x+1 \\ c|p^d\pm 1}} \varphi^+(c)\,\kappa(p,c)^{d/f^+(p,c)}. \tag{36}$$

Comparing (36) with (16), we see that

$$|\Psi(p^d, x, s)| \leqslant \Phi(p^d, x). \tag{37}$$

This relation will largely reduce the proof to our previous estimates. Indeed, as in (17), we can write

$$\log \prod_{c\leqslant x+1} L_c(s) = \sum_{\substack{p^d>x \\ d\geqslant 2}} \Psi(p^d, x, s) + \sum_{p>x} \Psi(p, x, s) + \sum_{p^d\leqslant x} \Psi(p^d, x, s),$$

and the first and second sums on the right-hand side are $o(1)$ by (18), (19), and (37). Thus to prove the theorem it suffices to show that

$$\sum_{p^d\leqslant x} \Psi(p^d, x, s) = \log \zeta(s)^{-1} + o(1). \tag{38}$$

The argument will be similar to the argument for (20).

The first point is that the condition $p^d \leqslant x$ in (38) renders the condition $c \leqslant x+1$ superfluous in (36). Thus in the context of (38) we have

$$\Psi(p^d, x, s) = \frac{p^{-ds}}{d} \sum_{c|p^d \pm 1} \varphi^+(c)\, \kappa(p,c)^{d/f^+(p,c)}, \tag{39}$$

Now suppose that $c|p^d \pm 1$ (in other words that $f^+(p,c)|d$) and that $c \geqslant 3$. The conditions $p^d \equiv -1 \bmod c$ and $\kappa(p,c)^{d/f^+(p,c)} = -1$ are equivalent, because both are equivalent to the assertion that $f(p,c) = 2f^+(p,c)$ and $d/f^+(p,c)$ is odd. It follows that the complementary conditions, namely $p^d \equiv 1 \bmod c$ and $\kappa(p,c)^{d/f^+(p,c)} = 1$, are also equivalent, whence

$$\sum_{\substack{c \geqslant 3 \\ c|p^d \pm 1}} \varphi^+(c)\, \kappa(p,c)^{d/f^+(p,c)} = \frac{1}{2} \sum_{\substack{c \geqslant 3 \\ c|p^d - 1}} \varphi(c) - \frac{1}{2} \sum_{\substack{c \geqslant 3 \\ c|p^d + 1}} \varphi(c). \tag{40}$$

As before, the restriction $c \geqslant 3$ can be eliminated throughout (40), because if $c = 1$ or 2, then $\kappa(p,c) = 0$.

With the restriction $c \geqslant 3$ removed, (40) implies that

$$\sum_{c|p^d \pm 1} \varphi^+(c)\, \kappa(p,c)^{d/f^+(p,c)} = \frac{1}{2}((p^d - 1) - (p^d + 1)) = -1,$$

and therefore (39) gives

$$\sum_{p^d \leqslant x} \Psi(p^d, x, s) = -\sum_{p \leqslant x} \sum_{d \leqslant \frac{\log x}{\log p}} \frac{p^{-ds}}{d}. \tag{41}$$

On the other hand,

$$\log \zeta(s)^{-1} = -\sum_{p} \sum_{d \geqslant 1} \frac{p^{-ds}}{d} \tag{42}$$

Taking the absolute value of the difference of (41) and (42), we find

$$\left| \sum_{p^d \leqslant x} \Psi(p^d, x, s) - \zeta(s)^{-1} \right| \leqslant \sum_{p^d > x} \frac{p^{-ds}}{d}.$$

Since $\Re(s) \geqslant 2$ the right-hand side is the tail of a convergent series (namely the Dirichlet series for $\log \zeta(s)$) and is therefore $o(1)$. Thus (38) follows and the proof of the theorem is complete.

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